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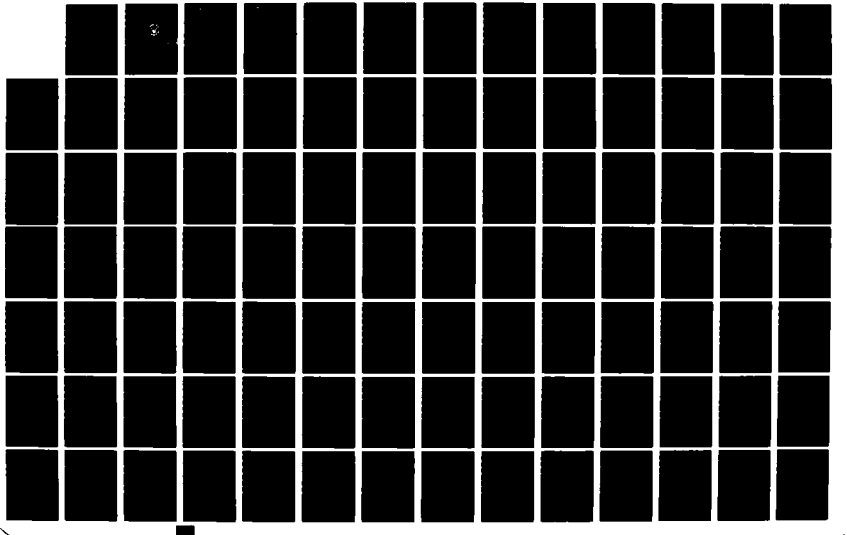

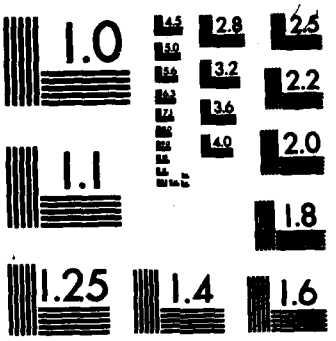

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THESIS

A NUMERICAL ANALYSIS OF PIPE FLOW STABILITY

by

David Bruce Wallace

December 1982

Thesis Advisor: T. H. Gawain

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Standard theoretical methods of analysis which work well for seemingly more complex problems fail to predict the experimentally observed instability of fully developed, incompressible pipe flow at any Reynolds number. Past research by Harrison and Arnold on the stability of pipe flow yielded erroneous results due to errors in the setup of the problem and formulation of the boundary conditions at the axis.

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20. ABSTRACT (Continued)

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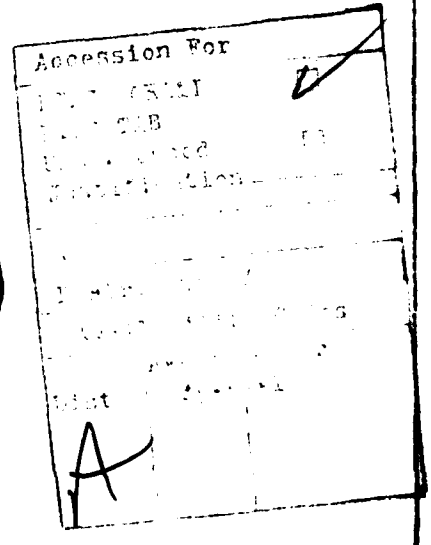
Numerical results for angular wave numbers zero and six show that the flow is stable at all Reynolds numbers. Results for angular wave number one contain instabilities at all Reynolds numbers for small values of the axial wave number. These results are tabulated, plotted, and discussed in detail in this paper.



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A Numerical Analysis of Pipe Flow Stability

by

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Lieutenant, United States Navy
B.S., University of Kansas, 1975

Submitted in partial fulfillment of the requirements
for the degree of

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ABSTRACT

Standard theoretical methods of analysis which work well for seemingly more complex problems fail to predict the experimentally observed instability of fully developed, incompressible pipe flow at any Reynolds number. Past research by Harrison and Arnold on the stability of pipe flow yielded erroneous results due to errors in the setup of the problem and formulation of the boundary conditions at the axis.

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TABLE OF SYMBOLS

D	Differential operator $\partial/\partial r$
D^2, D^3, D^4	Higher order partial derivatives with respect to r.
e	Base of natural logrithms.
$\vec{e}_x$, $\vec{e}_r$, $\vec{e}_\theta$	Unit vectors along the x, r, and θ axes in cylindrical coordinates.
F, G, H	Components of the velocity vector potential function defined in equation (2-28).
h	The increment of the radius in the finite difference equations, $h = 1/N$.
i	$+\sqrt{-1}$, the imaginary unit.
i	Used as an index to distinguish the transformed coefficient matrices $[M_i']$ of equation (2-43) and in the discrete finite difference equations.
j	Used as an index to distinguish the transformed coefficient matrices $[N_j']$ of equation (2-43).
N	The number of interior points along the pipe radius used in the finite difference mesh.
n	Angular wave number of the perturbation in the θ direction, where $n = 0, 1, 2, 3 \ldots\ldots$
O	Symbol denoting the phrase "of order".
P	Pressure appearing in equations (2-4, 5, 11, and 12).

P	Component of the velocity vector potential derived from the component G after a change of variable.
Q	Component of the velocity vector potential derived from the component H after a change of variable.
R	Normalized pipe radius, $R = 1$.
$\bar{R}$	The dimensional pipe radius used to define Re in equation (2-6).
Re	Reynolds number based on pipe radius and mean volumetric velocity, as defined in equation (2-6).
t	Time.
U	The axial component of the mean dimensionless velocity of the flow as defined in equation (2-13).
$\bar{U}$	The mean dimensional volumetric velocity of the flow used to define Re in equation (2-6).
u, v, w	Components of the complex perturbation velocity.
$\vec{V}$	Velocity vector of the mean flow.
$\vec{V}'$	Velocity vector of the total or resultant flow.
$\vec{v}$	Complex perturbation velocity vector.
$\vec{W}$	The velocity vector potential function defined in equation (2-28).
x, r, θ	Cylindrical coordinates.
X	Symbol for $i\alpha x + in\theta + \gamma t$, defined in equation (2-29).

$\{X^*\}$	Eigenvector corresponding to the least stable eigenvalue.
α	The real axial wave number of the perturbation in the x direction, where $\alpha \geq 0$.
γ	The complex frequency of the perturbation defined in equation (2-30).
γ^*	The maximum algebraic value of the real part of the complex frequency which is also the least stable eigenvalue, also referred to as GAMMA*.
$\vec{\Gamma}$	Shorthand notation of the vorticity transport equations as defined in equation (2-32).
Γ_x, Γ_r, Γ_θ	Components of $\vec{\Gamma}$ in cylindrical coordinates.
ε	The residual error of the least stable eigenvalue solution of the governing equations.
υ	Kinematic viscosity.
μ	Viscosity.
ρ	Density.
$\vec{\Omega}$	Mean vorticity vector.
$\vec{\Omega}'$	Total or resultant vorticity vector.
$\vec{\omega}$	Perturbation vorticity vector.
∇	Linear vector operator (nabla).
$\times$	Vector cross-product operator.
$\nabla\times$	Vector curl operator.

$\nabla\cdot$	Vector dot product operator.
[]	Brackets enclosing a matrix.
{}	Brackets enclosing a column vector.

I. INTRODUCTION

Osborne Reynolds [Ref. 1] conducted his classical experiments on the transition from laminar to turbulent flow of fluids in circular pipes nearly 100 years ago. Based on his experiments the critical Reynolds number for pipe flow was found to be 2300. An analytical solution for the problem of predicting instabilities of fully developed, three dimensional, incompressible flow of constant viscosity fluids in pipes has been pursued actively since then. Several approaches have been taken in solving the inherently nonlinear Navier-Stokes equations, which along with the continuity equation, govern the behavior of fluid flow in pipes of circular cross-section.

Salwen and Grosch [Ref. 2] studied pipe flow with purely sinusoidal streamwise (axial) perturbations. Infinitesimal velocity and pressure perturbations, which were explicit functions of time and not complex or exponential in form, were introduced into the flow field. Numerical calculations were carried out at various angular wave numbers and it was concluded that the flow was stable for all axial wave numbers and Reynolds numbers. Perturbations with exponential growth in space but a purely sinusoidal variation in time were explored by Garg and Rouleau [Ref. 3] and found to be stable. Gill [Ref. 4] studied combinations of exponential growth in space and in time using power series analysis and concluded that these flows were all stable.

Because of the general agreement that pipe flow is stable to infinitesimal disturbances, two other approaches to the problem of predicting instabilities at the experimentally observed critical Reynolds number have been pursued. First, investigations by Davy and Drazin [Ref. 5] confirmed that pipe Poiseuille flow was stable at all Reynolds numbers for infinitesimal disturbances that were both temporally and axially complex and exponential in nature. However, despite the fact that the governing equations must necessarily remain nonlinear with the introduction of disturbances of finite amplitude, they concluded that the flow is unstable with respect to finite disturbances. Similar theoretical examinations of plane Poiseuille flow between two parallel plates led McIntire and Lin [Ref. 6] to this same conclusion. Second, it was postulated by Huang and Chen [Ref. 7] and Leite [Ref. 8] that the origin of flow instabilities occur in the entrance region of the pipe, where the flow has not yet become fully developed. Both theoretical and experimental studies have been done to support these hypotheses. Garg [Ref. 9] also found that flow instabilities existed near the entry region for developing pipe flow. He concluded that results using the Hornbeck velocity profile more nearly approximated experimentally determined values of the critical Reynolds number. While these investigations have shown instabilities to exist in pipe flow, a completely generalized solution to the linearized problem of fully developed pipe flow has never been accomplished.

Recently, a more general theory consisting of perturbations which have a fully complex exponential form with respect to time and the axial

coordinate, a purely imaginary exponential form in the angular coordinate and are three dimensional in nature have been explored. Development of this theory by Harrison [Ref. 10] and further numerical investigations by Arnold [Ref. 11] failed to produce conclusive results that instabilities exist in fully developed pipe flow. Errors in the setup of the problem and inadequate formulation of the boundary conditions at the pipe axis contributed to these erroneous results. Arnold, however, was on the right track in his research by incorporating newly formulated boundary conditions at the axis developed by Gawain [Ref. 12].

Arnold's work, for the most part, remains valid except that he overlooked a small but significant detail that resulted in unclear definitions of stability and instability. He used a complex axial wave number α, of the form $\alpha = \alpha_R + i\alpha_I$, whereas the correct version should simply be a purely imaginary quantity of the form $i\alpha$. Thus, Arnold introduced an extra degree of freedom, namely the fictitious and incorrect quantity α_R. This accounts for the erroneous instabilities which he obtained in his numerical investigation of pipe flow.

Further work by Gawain [Ref. 13] refined the boundary conditions at the axis and incorporated the previously defined imaginary axial wave number, in exponential form, which represents only sinusoidal oscillations with respect to the axial coordinate. Using these advancements in the linearized theory, Gawain showed an asymptotic trend toward neutral stability of the flow with increasing Reynolds numbers for angular wave number, $n = 0$. This paper duplicates and confirms the results for $n = 0$, that the flow is stable at all Reynolds numbers and axial wave numbers. It is further shown that fully developed pipe flow for $n = 6$

is also stable and shows the same trends as for n = 0. Instabilities are shown to exist in the numerical analysis of the vorticity transport equations for n = 1. The paradox that now appears is that for angular wave numbers n = 0 and 6 the flow is apparently stable at all Reynolds numbers while for n = 1 the flow is apparently unstable at all Reynolds numbers. Neither of these theoretical results is consistent with the known experimental fact that there exists a critical Reynolds number below which pipe flow is stable and above which it is unstable. However, the new result for n = 1, while it can hardly be called correct, is at least encouraging in that it shows for the first time that instabilities can in fact exist in the solution of the linearized vorticity transport equations for fully developed pipe flow.

In addition to implementing the improved axis boundary conditions and the advancement in the linearized theory, improved finite differencing techniques were used to approximate the vorticity transport equations along a uniform radial mesh. Highly accurate numerical solutions for the previously mentioned angular wave numbers, n = 0, 1, and 6, were obtained using double precision, complex numbers. Additionally, finite difference equations with consistent fourth order truncation error were used throughout. Details of the development of the vorticity transport equations and the central and non-central finite difference equations are discussed in later chapters. Systematic and extensive calculations remain to be accomplished for angular wave numbers, n = 2, 3, 4, 5, and n > 6. The results of these numerical analyses should prove to be fruitful in establishing a theory which

adequately explains the experimentally observed fact of the onset of flow instabilities at the critical Reynolds number.

II. THE VORTICITY TRANSPORT EQUATION

A. BASIC THEORY

The governing equations for laminar flow of fluids in a circular pipe are derived from Newton's second law of motion,

$$\vec{F} = m\,\vec{a} \tag{2-1}$$

and the conservation of mass principle through an infinitesimal control volume. Equation (2-1) results in the familiar Navier-Stokes equation, which is a vector equation in three cylindrical coordinates and in time. The equation for conservation of mass results in the continuity equation for incompressible flow and is given by,

$$\nabla \cdot \vec{V} = 0 \tag{2-2}$$

For the case at hand, the following assumptions are made,

1. The fluid is incompressible, ρ = constant
2. The fluid has constant viscosity, μ = constant
3. The flow is fully developed, $U = f(r)$ only
4. The flow is three dimensional in nature
5. The mean flow is steady, $\partial\vec{V}/\partial t = 0$
6. The perturbation flow is unsteady, $\partial\vec{v}/\partial t \neq 0$
7. The effects of body forces are negligible

The resulting Navier-Stokes equation can be stated as the sum of the forces per unit mass is equal to the acceleration or,

$$\Sigma \frac{\text{Force}}{\text{mass}} = \text{acceleration} \qquad (2\text{-}3)$$

The forces acting on the fluid are pressure forces and viscous forces. Body forces and hydrostatic forces balance out and are not a factor here.

Equation (2-3) in its dimensional form can be expressed as,

$$- \nabla^* \frac{p^*}{\rho^*} + \left(\frac{\mu^*}{\rho^*}\right) \nabla^{2*} \vec{V}^* = \frac{d\vec{V}^*}{dt^*} \qquad (2\text{-}4)$$

where the starred quantities are fully dimensional.

It should be noted here that the equations presented through equation (2-18) represent the mean flow. Equation (2-4) can also be expressed in a nondimensional form as,

$$- \nabla P + \frac{1}{Re} \nabla^2 \vec{V} = \frac{d\vec{V}}{dt} \qquad (2\text{-}5)$$

where Re is the Reynolds number and is defined as,

$$Re = \frac{\bar{R}\,\bar{U}}{\nu} \qquad (2\text{-}6)$$

This definition of Reynolds number is based on the pipe radius $\bar{R}$ as the characteristic length, the mean volumetric velocity $\bar{U}$ as the characteristic velocity, and the kinematic viscosity ν which is constant. The three quantities on the right side of equation (2-6) are dimensional, but Re is a dimensionless quantity. On this basis, the critical Reynolds number for transition from laminar to turbulent flow becomes 1150 (vice the value of 2300 which is obtained if the Reynolds number is

defined in terms of pipe diameter). All subsequent equations presented in this paper are in nondimensional form.

Returning to the derivation of the mean flow, the acceleration term or substantial derivative, $d\vec{V}/dt$ can be expanded to the form,

$$\frac{d\vec{V}}{dt} = (\vec{V} \cdot \nabla)\,\vec{V} + \frac{\partial \vec{V}}{\partial t} \tag{2-7}$$

but $(\vec{V} \cdot \nabla)\vec{V}$ can be further expanded by a well known vector identity to,

$$(\vec{V} \cdot \nabla)\vec{V} = \nabla\,(\frac{U^2}{2}) - \vec{V} \times (\nabla \times \vec{V}) \tag{2-8}$$

Additionally, $\nabla^2\vec{V}$ from equation (2-5) can be expanded to,

$$\nabla^2\,\vec{V} = \nabla\,(\nabla \cdot \vec{V}) - \nabla \times (\nabla \times \vec{V}) \tag{2-9}$$

but $\nabla \cdot \vec{V} = 0$, since continuity must be satisfied and equaton (2-9) becomes,

$$\nabla^2\,\vec{V} = -\,\nabla \times (\nabla \times \vec{V}) \tag{2-10}$$

With appropriate substitutions, the final form of the Navier-Stokes equation becomes,

$$-\nabla P - \frac{1}{Re}\,\nabla \times (\nabla \times \vec{V}) = \nabla\,(\frac{U^2}{2}) - \vec{V} \times (\nabla \times \vec{V}) + \frac{\partial \vec{V}}{\partial t} \tag{2-11}$$

A more convenient form of equation (2-11) is,

$$-\nabla\,(P + (\frac{U^2}{2})) = \frac{1}{Re}\,\nabla \times (\nabla \times \vec{V}) - \vec{V} \times (\nabla \times \vec{V}) + \frac{\partial \vec{V}}{\partial t} \tag{2-12}$$

As previously discussed, all the quantities in the equations have been nondimensionalized. Dimensionless cylindrical coordinates x, r, and θ are used with $\vec{e}_x$, $\vec{e}_r$ and $\vec{e}_\theta$ denoting the unit vectors in the three coordinate directions, respectively. It is well known that the velocity profile for fully developed, three dimensional, incompressible flow in circular pipes can be expressed analytically by,

$$U = 2\,(1 - r^2) \tag{2-13}$$

This function has the shape of a paraboloid of revolution as shown in Figure 2-1.

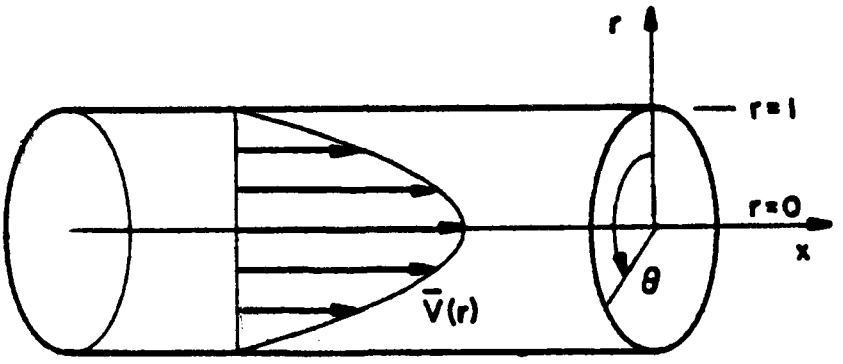


Figure 2-1. Velocity Profile of Fluid Flow in a Pipe

The nondimensional mean velocity vector $\vec{V}(r)$ of the flow is given by,

$$\vec{V}(r) = U\,\vec{e}_x = 2\,(1 - r^2)\,\vec{e}_x \tag{2-14}$$

A second vector quantity $\vec{\Omega}$, the mean vorticity, is defined as the curl of the mean velocity vector and is given by,

$$\vec{\Omega} = \nabla \times \vec{V} = 4\, r\, \vec{e}_\theta \tag{2-15}$$

which will be used later.

B. THE VORTICTY TRANSPORT EQUATION

The vorticity transport equation is now obtained by taking the curl of the Navier-Stokes equation (2-12), resulting in,

$$- \nabla \times \nabla \left(P + \left(\frac{U^2}{2}\right)\right) = \frac{1}{Re} \nabla \times (\nabla \times (\nabla \times \vec{V})) - \nabla \times (\vec{V} \times (\nabla \times \vec{V})) + \frac{\partial(\nabla \times \vec{V})}{\partial t} \tag{2-16}$$

The primary reason for taking the curl of the Navier-Stokes equation is to eliminate the unknown scalar on the left side of equation (2-16). Since the curl of the gradient of a scalar is equal to zero, equation (2-6) reduces to,

$$\frac{1}{Re} \nabla \times (\nabla \times (\nabla \times \vec{V})) - \nabla \times (\vec{V} \times (\nabla \times \vec{V})) + \frac{\partial(\nabla \times \vec{V})}{\partial t} = 0 \tag{2-17}$$

Equation (2-17) can be simplified by substituting the mean vorticity vector relation equation (2-15) into equation (2-17) resulting in,

$$\frac{1}{Re} \nabla \times (\nabla \times \vec{\Omega}) - \nabla \times (\vec{V} \times \vec{\Omega}) + \frac{\partial \vec{\Omega}}{\partial t} = 0 \tag{2-18}$$

In order to analyze this flow field and in particular to determine the transition from laminar to turbulent flow, a disturbance of small amplitude is introduced into the flow. It is assumed that the resulting flow is made up of the steady state, laminar flow and a small perturbation flow superimposed on the laminar flow. The total velocity and vorticity respectively become,

$$\vec{V}' = \vec{V} + \vec{v} \tag{2-19}$$

and

$$\vec{\Omega}' = \vec{\Omega} + \vec{\omega} \tag{2-20}$$

where $\vec{v}$ and $\vec{\omega}$ are the velocity perturbations and vorticity perturbations, respectively. It is of course a necessary requirement that the continuity equation for the total flow be satisfied here as well,

$$\nabla \cdot \vec{V}' = \nabla \cdot \vec{V} \quad + \quad \nabla \cdot \vec{v} = 0 \tag{2-21}$$

In view of this relation and of equation (2-2), the perturbation flow independently satisfies the continuity condition and

$$\nabla \cdot \vec{v} = 0 \tag{2-22}$$

The introduction of the perturbation quantities into the vorticity transport equation is accomplished by substituting $\vec{V}'$ for $\vec{V}$ and $\vec{\Omega}'$ for $\vec{\Omega}$ into equation (2-18). This results in,

$$\frac{1}{Re} \nabla \times (\nabla \times (\vec{\Omega} + \vec{\omega})) - \nabla \times [(\vec{V} + \vec{v}) \times (\vec{\Omega} + \vec{\omega})] + \frac{\partial(\vec{\Omega} + \vec{\omega})}{\partial t} = 0 \tag{2-23}$$

which is the equation for the total or resultant flow field. Upon expanding equation (2-23), applying the mean steady flow assumption, $\partial\vec{V}/\partial t = 0$, and subtracting equation (2-18) from it, equation (2-23) becomes,

$$\frac{1}{Re}\nabla\times(\nabla\times\vec{\omega}) - \nabla\times[(\vec{V}\times\vec{\omega}) - (\vec{\Omega}\times\vec{v}) + (\vec{v}\times\vec{\omega})] + \frac{\partial\vec{\omega}}{\partial t} = 0 \tag{2-24}$$

Now equation (2-23) is linear in the perturbation quantities except for the second order term, $(\vec{v}\times\vec{\omega})$. Since the perturbations are assumed to be small, this term can be neglected.

The perturbation vorticity is defined as,

$$\vec{\omega} = \nabla\times\vec{v} \tag{2-25}$$

which is analogous to equation (2-15) for the mean flow. Making this substitution in equation (2-24) yields the linearized vorticity transport equation,

$$\frac{1}{Re}\nabla\times(\nabla\times(\nabla\times\vec{v})) - \nabla\times(\vec{V}\times(\nabla\times\vec{v})) + \nabla\times(\nabla\times(\vec{V}\times\vec{v})) + \nabla\times\frac{\partial\vec{v}}{\partial t} = 0 \tag{2-26}$$

As previously mentioned, the continuity equation (2-22) for the perturbation velocity must be satisfied. A convenient way to insure this is to express $\vec{v}$ in terms of a velocity vector potential function $\vec{W}$ as follows,

$$\vec{v} = \nabla \times \vec{W} \tag{2-27}$$

It can be verified that if $\vec{v}$ is defined in this way, equation (2-22) is satisfied identically for any arbitrary vector function $\vec{W}$.

Instead of proceeding directly with the real vector functions $\vec{v}$ and $\vec{W}$, it is advantageous to work with the Fourier transforms of $\vec{v}$ and $\vec{W}$, which are in general complex functions. This is the form which is used here. It can be shown, because of the linearity of the basic equation, that if a solution can be found for the Fourier transforms, then the corresponding real vector functions will satisfy the basic equation as well. A detailed explanation of these functions was presented more elegantly by Gawain [Ref. 13]. The vector potential function $\vec{W}$, now representing the Fourier transform, can be expressed as,

$$\vec{W} = [F(r)\ \vec{e}_x + G(r)\ \vec{e}_r + H(r)\ \vec{e}_\theta]\ e^X \tag{2-28}$$

where X is a convenient abbreviation for the complex quantity,

$$X = i\alpha x + in\theta + \gamma t \tag{2-29}$$

The axial wave number α is a real quantity greater than or equal to zero which in the exponential form represents purely sinusoidal axial variations of $\vec{v}$. The angular wave number n is a positive integer and represents sinusoidal angular variations of $\vec{v}$ which are periodic with respect to coordinate θ. The complex frequency γ contained in the exponent is of the form,

$$e^{\gamma t} = e^{(\gamma_R + i\gamma_I)t} = e^{\gamma_R t} (\cos \gamma_I t + i \sin \gamma_I t) \qquad (2\text{-}30)$$

The quantity γ_R clearly represents the exponential time rate of growth or decay of the perturbation amplitude. The basic criteria for hydrodynamic stability can be defined in terms of γ_R. Positive values of γ_R represent growth and signify flow instability while negative values of γ_R represent decay and signify flow stability.

The vorticity transport equation can now be expressed in terms of the velocity vector, $\vec{V}$ and the complex vector potential, $\vec{W}$. By making the substitution for $\vec{v}$, equation (2-26) now becomes,

$$\frac{1}{Re} \nabla \times (\nabla \times (\nabla \times (\nabla \times \vec{W}))) - \nabla \times (\vec{V} \times (\nabla \times (\nabla \times \vec{W})))$$

$$+ \nabla \times (\nabla \times (\vec{V} \times (\nabla \times \vec{W}))) + \nabla \times (\nabla \times \frac{\partial \vec{W}}{\partial t}) = 0 \qquad (2\text{-}31)$$

In convenient shorthand notation, equation (2-31) becomes,

$$\text{Equation (2-31)} = \vec{\Gamma}(r)e^{X} = [\Gamma_x(r)\, \vec{e}_r + \Gamma_r(r)\, \vec{e}_\theta + \Gamma_\theta(r)\, \vec{e}\,]\, e^{X} = 0 \qquad (2\text{-}32)$$

and can be expressed in matrix notation as,

$$\vec{\Gamma} = \begin{Bmatrix} \Gamma_x \\ \Gamma_r \\ \Gamma_\theta \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \\ 0 \end{Bmatrix} \qquad (2\text{-}33)$$

Therefore, the three components of the vorticity transport equation are equivalent to three simultaneous scalar equations, namely,

$$\Gamma_x = 0$$

$$\Gamma_r = 0 \qquad (2\text{-}34)$$

$$\Gamma_\theta = 0$$

Of these three equations, only two are actually independent since, as can be seen from equation (2-31), $\vec{\Gamma}\, e^x$ is itself the curl of another vector. Therefore $\vec{\Gamma}\, e^x$ must necessarily be nondivergent. It can be shown that the components of $\vec{\Gamma}$ satisfy the identity, $\nabla \cdot (\vec{\Gamma}\, e^x) \equiv 0$, or,

$$i\alpha\Gamma_x + \frac{1}{r} D\,(r\Gamma_r) + \left(\frac{in}{r}\right)\Gamma_\theta \equiv 0 \qquad (2\text{-}35)$$

where D is the symbol for the differential operator d/dr. Two linearly independent equations can now be formulated from equations (2-34) given the constraint of equation (2-35). This can be reduced to some appropriate linear combination of two of the equations which is set equal to zero and to the third equation which is also set equal to zero. This results in a system of two coupled, fourth order differential equations in three unknowns. Recall that the vector potential function $\vec{W}$ is expressed in terms of F(r), G(r), and H(r). It can be shown that one of these variables is redundant and may be eliminated without any loss of generality in the solution of the equations. For the present case, the solution assumes its most convenient form if,

$$F(r) = 0 \qquad (2\text{-}36)$$

This convention has therefore been adopted and now G(r) and H(r) become the two coupled eigenfunctions of the governing fourth order differential equations. Returning to the problem of selecting the proper linear combination of equations comprising $\vec{\Gamma}$, it is important to choose this combination in an optimum fashion by analyzing the algebraic structure of these equations. Table 2-1 is taken from reference [13], corrected, and duplicated here for that purpose. Notice that equations (1) and (3) in the table are third order in G. If the particular linear combination of these equations is chosen as appears in equation (4), the result reduces to an equation of second order in G. This does not affect the order of the result since the function H remains fourth order.

Table 2-1. Properties of the Vorticity Transport Equations

EQUATION	TERM OF HIGHEST ORDER IN G	TERM OF HIGHEST ORDER IN H	HIGHEST NEGATIVE POWER OF r
(1) $\Gamma_x(r) = 0$	$\frac{i\alpha}{Re} D^3G$	$-\frac{n\alpha}{rRe} D^2H$	r^{-3}
(2) $\Gamma_r(r) = 0$	$\frac{1}{Re}(\alpha^2 + \frac{n^2}{r^2}) D^2G$	$\frac{in}{rRe} D^3H$	r^{-4}
(3) $\Gamma_\theta(r) = 0$	$\frac{in}{rRe} D^3G$	$-\frac{1}{Re} D^4H$	r^{-4}
(4) $-\frac{n}{\alpha r}\Gamma_x(r)+\Gamma_\theta(r)=0$	$-\frac{4\,in}{r^2Re} D^2G$	$-\frac{1}{Re} D^4H$	r^{-4}

The formulation of the problem is simplified greatly by choosing two independent vorticity transport equations in the form,

$$\Gamma_r = 0 \tag{2-37}$$

and

$$-\frac{n}{\alpha r}\Gamma_x + \Gamma_\theta = 0 \tag{2-38}$$

C. THE VORTICITY TRANSPORT MATRIX EQUATION

While the fourth order linear vector operations and the associated algebra becomes very lengthy and rather tedious, the investigator attempting to expand the final form of the linearized vorticity transport equation (2-31) should pay sufficient attention to these details. Recall that the curl operation on a vector in cylindrical coordinates takes the form,

$$\nabla \times \vec{W} = \begin{vmatrix} \frac{1}{r}\vec{e}_x & \frac{1}{r}\vec{e}_r & \vec{e}_\theta \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial r} & \frac{\partial}{\partial \theta} \\ 0 & Ge^x & rHe^x \end{vmatrix} \tag{2-39}$$

After all the nabla (∇) operations are complete, the final equation will be in the form of a vector equation as defined in equation (2-32). The first vorticity transport equation consists of the coefficient represented by Γ_r and is set equal to zero. The second equation consists of

the previously defined linear combination of the coefficients represented by Γ_x and Γ_θ and is also set equal to zero. The two resulting simultaneous ordinary differential equations are then expanded and the coefficients of the derivatives regrouped so that equations (2-37) and (2-38) can be conveniently expressed in matrix format as follows,

$$[M_4]\begin{Bmatrix} D^4G \\ D^4H \end{Bmatrix} + [M_3]\begin{Bmatrix} D^3G \\ D^3H \end{Bmatrix} + [M_2]\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [M_1]\begin{Bmatrix} DG \\ DH \end{Bmatrix}$$

$$+ [M_0]\begin{Bmatrix} G \\ H \end{Bmatrix} = \gamma\left([N_2]\begin{Bmatrix} D^2G \\ D^2H \end{Bmatrix} + [N_1]\begin{Bmatrix} DG \\ DH \end{Bmatrix} + [N_0]\begin{Bmatrix} G \\ H \end{Bmatrix}\right) \qquad (2\text{-}40)$$

As it turns out, the equations are a pair of coupled homogeneous differential equations that can be solved as an eigenvalue problem. This technique is discussed in the chapter on Numerical Methods.

The matrices which appear in equation (2-40) are 2 X 2 matrices and are summarized in detail in Appendix A. Equation (2-40) and the respective matrix elements contained in Appendix A are the generalized vorticity transport equations for all angular wave numbers n, at all axial wave numbers α, and all Reynolds numbers. As will be explained in detail in the chapter on Boundary Conditions, the generalized equations are transformed by appropriate changes of variable depending on the angular wave number. The changes of variables from the eigenfunctions G(r) and H(r) to the new eigenfunctions P(r) and Q(r), respectively, are determined by the original boundary conditions that must be satisfied, particularly at the pipe axis.

D. EQUATIONS FOR n = 0

For the case n = 0, equation (2-40) is greatly simplified in that the two equations uncouple and allow an independent investigation of either eigenfunction, G(r) or H(r). Miscellaneous trial calculations made earlier suggest that the solutions for G(r) are probably stable for all values of α and Re. Therefore, only the solutions for H(r) were investigated. It should also be noted that the examination of the axial perturbation velocity is of particular interest in this analysis and, in the case for n = 0, is not a function of G(r). The change of variable required to satisfy the boundary conditions is,

$$H(r) = r\,Q(r) \tag{2-41}$$

Taking the derivatives of H(r) yields,

$$\begin{aligned} DH &= r\,DQ + Q \\ D^2H &= r\,D^2Q + 2\,DQ \\ D^3H &= r\,D^3Q + 3\,D^2Q \\ D^4H &= r\,D^4Q + 4\,D^3Q \end{aligned} \tag{2-42}$$

With the substitution of equations (2-41) and (2-42) into equation (2-40), the following expression is obtained,

$$M_4' D^4Q + M_3' D^3Q + M_2' D^2Q + M_1' DQ + M_0' Q = \gamma (N_2' D^2Q + N_1' DQ + N_0' Q) \tag{2-43}$$

The primed coefficients are actually the elements from row 2 and column 2 of the newly transformed coefficient matrices and are not enclosed in matrix brackets. However, in general, the new coefficient matrices $[M_i']$ and $[N_j']$, (where the indices i = 0, 1, 2, 3, 4 and j = 0, 1, 2), are formed after the change of variable is made and the terms are collected. The primed coefficients used to solve the transformed vorticity transport equation in terms of Q(r) for n = 0 are defined in Appendix B, Part A.

E. EQUATIONS for n = 1

For the case n = 1, eigenfunctions G(r) and H(r) do not become uncoupled and must be solved for in a system of simultaneous equations in a form similar to equation (2-40). An additional parameter H(0) is introduced into the equations by the change of variables. This particular change of variables is required because of the complicated nature of the boundary conditions at the axis and is thoroughly explained in the chapter on Boundary Conditions. For n = 1, the change of variables required to satisfy the boundary conditions are,

$$\begin{aligned} G(r) &= -i\, H(0) + r^2\, P(r) \\ H(r) &= H(0) + r^2\, Q(r) \end{aligned} \qquad (2\text{-}44)$$

Taking the derivatives of G(r) yields,

$$\begin{aligned} DG &= r^2\, DP + 2r\, P \\ D^2G &= r^2\, D^2P + 4r\, DP + 2\, P \end{aligned} \qquad (2\text{-}45)$$

and taking the derivatives of H(r) yields,

$$\begin{aligned} DH &= r^2\, DQ + 2r\, Q \\ D^2H &= r^2\, D^2Q + 4r\, DQ + 2\, Q \\ D^3H &= r^2\, D^3Q + 6r\, D^2Q + 6\, DQ \\ D^4H &= r^2\, D^4Q + 8r\, D^3Q + 12\, D^2Q \end{aligned} \tag{2-46}$$

Remember that in general $D(H(0)) \neq DH(0)$, where $DH(0)$ is the value of the first derivative of H(r) evaluated at the point r = 0. Since $D(H(0)) = 0$ here, it vanishes from the expressions for the derivatives of G(r) and H(r). In order to accommodate H(0) in the system of equations where it appears explicitly, two additional 2 X 1 column matrices are required, namely $[M_5']$ and $[N_3']$. After the changes of variable have been made, equation (2-40) now becomes,

$$[M_4'] \begin{Bmatrix} D^4P \\ D^4Q \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3P \\ D^3Q \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [M_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [M_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix}$$

$$+ [M_5'] \{H(0)\} = \gamma \left([N_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [N_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [N_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} + [N_3'] \{H(0)\} \right) \tag{2-47}$$

The coefficient matrices of equation (2-47) used to solve the transformed vorticity transport equations in terms of P(r) and Q(r) for n = 1 are defined in Appendix B, Part B.

F. EQUATIONS FOR n = 6

For the case n = 6, eigenfunctions G(r) and H(r) remain coupled. The change of variables does not introduce any additional parameters and the system of simultaneous equations is solved in the form of equation (2-40) once again. The change of variables required to satisfy the boundary conditions for n = 6 and additionally for all angular wave numbers n > 6 are,

$$G(r) = r^4 P(r)$$
$$H(r) = r^3 Q(r) \qquad (2\text{-}48)$$

Taking the derivatives of G(r) yields,

$$DG = r^4 DP + 4r^3 P$$
$$D^2G = r^4 D^2P + 8r^3 DP + 12r^2 P \qquad (2\text{-}29)$$

and taking the derivatives of H(r) yields,

$$DH = r^3 DQ + 3r^2 Q$$
$$D^2H = r^3 D^2Q + 6r^2 DQ + 6r Q$$
$$D^3H = r^3 D^3Q + 9r^2 D^2Q + 18r DQ + 6 Q$$
$$D^4H = r^3 D^4Q + 12r^2 D^3Q + 36r D^2Q + 24 DQ \qquad (2\text{-}50)$$

Substituting equations (2-48), (2-49), and (2-50) into equation (2-40) results in the transformed coefficient matrices for the system of equations. The coefficient matrices of the functions P(r) and Q(r) and their respective derivatives for n = 6 are defined in Appendix B, Part C.

III. BOUNDARY CONDITIONS

The incorrect formulation of the boundary conditions for the linearized vorticity transport equations at the pipe axis has probably been the primary reason that previous investigators have failed to predict the onset of flow instabilities in circular pipes at any Reynolds number. Gawain [Ref. 13] suggested that if these boundary conditions are formulated in a rigorous and systematic fashion and applied to the vorticity transport equations, a correct numerical solution may be computed which agrees with experimental results. It can be seen from equations (2-40) and from the coefficient matrices of Appendix A that the two vorticity transport equations are coupled, except for the case $n = 0$. The resulting pair of differential equations are second order in $G(r)$ and fourth order in $H(r)$. In order to obtain a determinate solution of the equations, a total of six boundary conditions are required. More specifically, at least two of the boundary conditions must involve $G(r)$ or its derivatives and at least four of the boundary conditions must involve $H(r)$ or its derivatives. It might be expected that when the equations are coupled, the boundary conditions may also be coupled. That is to say that the boundary conditions may consist of various linear combinations of the functions and their derivatives, which turns out to be the case and is shown later.

The boundary conditions at the pipe wall are determined in a fairly straight forward manner. They are derived from the fact that the components of the perturbation velocity must vanish at the wall. From the definition of the perturbation velocity, defined previously as,

$$\vec{v} = \nabla \times \vec{W} \tag{2-27}$$

it can be shown that the following three boundary conditions result.

$$\begin{aligned} G(1) &= 0 \\ H(1) &= 0 \\ DH(1) &= 0 \end{aligned} \tag{3-1}$$

Since six boundary conditions are required and three have been determined at the pipe wall, there remain three boundary conditions to be determined at the pipe axis. Of these three conditions, at least one must involve G(r) or its derivatives and at least two must involve H(r) or its derivatives.

The proper derivation of the three boundary conditions at the axis turns out to be a non-trival problem. As can be seen in Table 2-1, the highest negative power of r that appears in the equations is r^{-4}. Closer inspection of the vorticity transport equations of equation (2-40) and the coefficient matrices in Appendix A reveals that the equations contain terms in r^{-4}, r^{-3}, r^{-2}, r^{-1}, and r^{0}. At first glance, it may appear that these equations are not satisfied in the limit as r approaches zero. It should be noted that it would be incorrect to multiply the equations through by any positive power of r since that would alter the degree of the singularity of the equations at the axis.

This problem can be resolved and subsequently the boundary conditions at the axis can be deduced rigorously by expanding the functions G(r) and H(r) as power series in r. The highest negative power of r is r^{-4} and appears in the coefficient matrix $[M_0]$ for the pair of functions G(r) and H(r). The series expansion for G(r), and similarly for H(r), is carried to the fourth derivative as follows,

$$G(r) = G(0) + DG(0)r + D^2G(0)\frac{r^2}{2!} + D^3G(0)\frac{r^3}{3!} + D^4G(0)\frac{r^4}{4!} + \ldots\ldots \quad (3\text{-}2)$$

It can be seen that when the series expansion expressions for G(r) and H(r) are substituted into equation (2-40), the higher order terms of equation (3-2) that contain powers of r greater than four will vanish in the limit as r approaches zero. For the first derivatives of G(r) and H(r), the highest negative power of r that appears in the coefficient matrix $[M_1]$ is r^{-3}. Therefore, the series expansion for the first derivative of G(r), and similarly for H(r), up to the fourth derivative term is,

$$DG(r) = 0 + DG(0) + D^2G(0)r + D^3G(0)\frac{r^2}{2!} + D^4G(0)\frac{r^3}{3!} + \ldots\ldots \quad (3\text{-}3)$$

The remaining derivatives of the functions are expanded in a similar manner through the fourth derivative term. The resulting power series approximations for G(r) and H(r) and their derivatives are then substituted into equation (2-40). The coefficients are then regrouped in ascending powers of r.

As mentioned previously, the equations must be satisfied in the limit as r approaches zero. Therefore, the coefficients of the functions that contain r^{-4}, r^{-3}, r^{-2}, r^{-1}, and r^{0} must all be zero. The remaining coefficients will contain positive powers of r and will vanish in the limit as r approaches zero. By considering only the terms containing negative powers of r and setting those coefficients equal to zero, five pair of equations in ten unknowns remain so that the boundary conditions can be determined. The unknowns are the functions of G(r) and H(r) and their first four derivatives evaluated at the pipe axis, r = 0. It is convenient to summarize these equations in matrix format below,

$$\frac{(n^2-1)}{Re}\,[C_1]\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-4}$$

$$\frac{n^2}{Re}\,[C_2]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-5}$$

$$\frac{(n^2-1)}{Re}\,[C_3]\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} + (i\alpha[C_4] - \gamma\,[D_4])\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-6}$$

$$\frac{n(n^2-4)}{Re}\,[C_5]\begin{Bmatrix} D^3G(0) \\ D^3H(0) \end{Bmatrix} + n\left(2i\alpha\left(1 - \frac{i\alpha}{Re}\right) + \gamma\right)[C_6]\begin{Bmatrix} DG(0) \\ DH(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix} \tag{3-7}$$

$$\frac{(n^2-9)}{Re}\,[C_7]\begin{Bmatrix} D^4G(0) \\ D^4H(0)\end{Bmatrix} + (i\alpha\,[C_8] - \gamma\,[D_8])\begin{Bmatrix} D^2G(0) \\ D^2H(0)\end{Bmatrix}$$

$$+ (i\alpha\,[C_9] - \gamma\alpha^2\,[D_9])\begin{Bmatrix} G(0) \\ H(0)\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0\end{Bmatrix} \qquad (3\text{-}8)$$

The 2 X 2 matrices which appear in equations (3-4) through (3-8) are defined in detail in Appendix C. Special conditions at the axis arise when the determinants of these matrices are evaluated at specific angular wave numbers. They are summarized in Appendix D.

For each specific value of angular wave number n, it is possible to derive a set of boundary conditions at the axis. Special conditions at the axis occur for angular wave numbers that cause the coefficients in equations (3-4) through (3-8) to equal zero or the determinants in Appendix D to be zero. That is to say, if a coefficient vanishes or the pair of equations are linearly dependent, the functions may be arbitrarily specified and are in general not equal to zero. But if the coefficient, appearing in equations (3-4) through (3-8) is non-zero and the determinant exists, the pair of functions must be identically zero. This situation occurs for all five pairs of variables when the angular wave number $n \geq 6$.

By taking these special conditions into account for the case n = 0, it can be deduced from equations (3-6) and (3-8) that,

$$\begin{Bmatrix} G(0) \\ H(0)\end{Bmatrix} = \begin{Bmatrix} D^2G(0) \\ D^2H(0)\end{Bmatrix} = \begin{Bmatrix} 0 \\ 0\end{Bmatrix} \qquad (3\text{-}9)$$

Carrying out this procedure for n = 1, 2, 3, 4, 5, and 6 yields a simplified set of boundary conditions at the axis. The following sets of equations (3-10) through (3-16) are essentially the same as those of Gawain [Ref. 13] but were independently checked and appear here with a few minor corrections.

$$
\begin{array}{l}
\underline{n = 0} \\
\qquad G(0) = 0 \qquad\qquad H(0) = 0 \qquad\qquad \text{(a)} \\
\qquad D^2G(0) = 0 \qquad\qquad D^2H(0) = 0 \qquad\qquad \text{(b)}
\end{array}
\tag{3-10}
$$

$$
\begin{array}{l}
\underline{n = 1} \\
\qquad G(0) + i\,H(0) = 0 \qquad\qquad \text{(a)} \\
\qquad DG(0) = 0 \qquad\qquad DH(0) = 0 \qquad\qquad \text{(a)} \\
\qquad D^3G(0) = 0 \qquad\qquad D^3H(0) = 0 \qquad\qquad \text{(a)} \\
-\frac{8}{Re}\,[C_7]\begin{Bmatrix} D^4G(0) \\ D^4H(0) \end{Bmatrix} + (i\alpha\,[C_8] - \gamma\,[D_8])\begin{Bmatrix} D^2G(0) \\ D^2H(0) \end{Bmatrix} \qquad \text{(b)} \\
+ (i\alpha\,[C_9] - \gamma\,\alpha^2\,[D_9])\begin{Bmatrix} G(0) \\ H(0) \end{Bmatrix} = \begin{Bmatrix} 0 \\ 0 \end{Bmatrix}
\end{array}
\tag{3-11}
$$

$$
\begin{array}{l}
\underline{n = 2} \\
\qquad G(0) = 0 \qquad\qquad H(0) = 0 \qquad\qquad \text{(a)} \\
\qquad DG(0) + i\,DH(0) = 0 \qquad\qquad \text{(a)} \\
\qquad D^2G(0) = 0 \qquad\qquad D^2H(0) = 0 \qquad\qquad \text{(b)} \\
\qquad D^4G(0) = 0 \qquad\qquad D^4H(0) = 0 \qquad\qquad \text{(b)}
\end{array}
\tag{3-12}
$$

$n = 3$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) + i\,D^2H(0) = 0 \qquad \text{(b)} \qquad \text{(3-13)}$$

$$D^3G(0) = 0 \qquad D^3H(0) = 0$$

$n = 4$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) = 0 \qquad D^2H(0) = 0$$

$$D^3G(0) + i\,D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-14)}$$

$$D^4G(0) = 0 \qquad D^4H(0) = 0$$

$n = 5$

$$G(0) = 0 \qquad H(0) = 0$$

$$DG(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^2G(0) = 0 \qquad D^2H(0) = 0$$

$$D^3G(0) = 0 \qquad D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-15)}$$

$$D^4G(0) + i\,D^4H(0) = 0$$

$n \geq 6$

$$G(0) = 0$$

$$DG(0) = 0 \qquad H(0) = 0$$

$$D^2G(0) = 0 \qquad DH(0) = 0 \qquad \text{(a)}$$

$$D^3G(0) = 0 \qquad D^2H(0) = 0$$

$$D^4G(0) = 0 \qquad D^3H(0) = 0 \qquad \text{(b)} \qquad \text{(3-16)}$$

$$D^4H(0) = 0$$

Recall that exactly three boundary conditions are required at the axis to finalize the solution of the vorticity transport equations. Equations (3-10) through (3-16) contain from four to ten boundary constraints including linear combinations of several boundary conditions, as suggested earlier. By introducing appropriate changes of variables, the pair of functions in G(r) and H(r) and their respective derivatives can be transformed to a new pair of functions in terms of P(r) and Q(r) and their derivatives. The transformed system of equations take the form of equation (2-40) for all values of n, except for n = 1 where equation (2-47) is applicable. The change of variables is dependent on each specific value of n up through six and is chosen in such a way that the boundary conditions in subset (a) of equation (3-10) through (3-16) are satisfied identically. The boundary conditions in subset (b) must be satisfied explicitly in the finite difference equations near the axis. Development of the finite difference equations is discussed in the chapter on Numerical Methods.

The following sets of equations are taken from Gawain [Ref. 13] and are included here so that a complete treatment of the subject on boundary conditions is contained in this paper.

$\underline{n = 0}$

Change of Variables

$$G(r) = rP(r)$$
$$H(r) = rQ(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-17)

$$DP(0) = 0 \qquad DQ(0) = 0$$
$$D^3Q(0) = 0 \qquad \text{(b)}$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$
$$DQ(1) = 0 \qquad \text{(c)}$$

The solutions for P(r) and Q(r) become uncoupled for n = 0

$\underline{n = 1}$

Change of Variables

$$G(r) = -i\,H(0) + r^2\,P(r)$$
$$H(r) = H(0) + r^2\,Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-18)

$$DP(0) = 0 \qquad DQ(0) = 0$$

$$-\frac{96}{Re}[C_7]\begin{Bmatrix} D^2P(0) \\ D^2Q(0) \end{Bmatrix} + 2i\alpha[C_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + i\alpha[C_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)$$

$$= +\gamma\left(2[D_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + \alpha^2\,[D_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)\right) \qquad \text{(b)}$$

Boundary Conditions at Wall

$$-i\,H(0) + P(1) = 0 \qquad H(0) + Q(1) = 0$$
$$-2H(0) + DQ(1) = 0 \qquad \text{(c)}$$

$n = 2$

Change of Variables

$$G(r) = -i\, r\, DH(0) + r^2\, P(r)$$

$$H(r) = r\, DH(0) + r^2\, Q(r) \qquad (a)$$

Boundary Conditions at Axis (3-19)

$$P(0) = 0 \qquad Q(0) = 0$$

$$D^2P(0) = 0 \qquad D^2Q(0) = 0 \qquad (b)$$

Boundary Conditions at Wall

$$-i\, DH(0) + P(1) = 0 \qquad DH(0) + Q(1) = 0$$

$$-DH(0) + DQ(1) = 0 \qquad (c)$$

$n = 3$

Change of Variables

$$G(r) = r^2\, P(r)$$

$$H(r) = r^2\, Q(r) \qquad (a)$$

Boundary Conditions at Axis (3-20)

$$P(0) + i\, Q(0) = 0$$

$$DP(0) = 0 \qquad DQ(0) = 0 \qquad (b)$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

$$DQ(1) = 0 \qquad (c)$$

---

n = 4

Change of Variables

$$G(r) = r^3 P(r)$$

$$H(r) = r^3 Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-21)

$$P(0) + i\,Q(0) = 0 \qquad \text{(b)}$$

$$DP(0) = 0 \qquad DQ(0) = 0$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0 \qquad \text{(c)}$$

$$DQ(1) = 0$$

---

n = 5

Change of Variables

$$G(r) = r^3 P(r)$$

$$H(r) = r^3 Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-22)

$$P(0) = 0 \qquad Q(0) = 0$$

$$DP(0) + i\,DQ(0) = 0 \qquad \text{(b)}$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0 \qquad \text{(c)}$$

$$DQ(1) = 0$$

$n \geq 6$

Change of Variables

$$G(r) = r^4 P(r)$$

$$H(r) = r^3 Q(r) \qquad \text{(a)}$$

Boundary Conditions at Axis (3-23)

$$P(0) = 0 \qquad Q(0) = 0$$

$$DQ(0) = 0 \qquad \text{(b)}$$

Boundary Conditions at Wall

$$P(1) = 0 \qquad Q(1) = 0$$

$$DQ(1) = 0 \qquad \text{(c)}$$

It should be noted that the change of variable relations in subset (a) of equations (3-17) through (3-23) does not change the order of the derivatives appearing in the transformed vorticity transport equations. Upon close examination of subsets (b) and (c) of these equations, it is apparent that the correct number of boundary conditions required for the solution of the transformed equations has resulted from the change of variables. The vorticity transport equations are now expressed in terms of P(r) and Q(r) and their derivatives and have exactly three boundary conditions at the pipe wall for all values of n and three boundary conditions at the axis for all values of n, except for n = 1 and 2. The fact that there are four boundary conditions at the axis for angular wave numbers n = 1 and 2 warrants further comment here.

Notice in equations (3-18a) and (3-19a) for the cases n = 1 and 2 that an additional parameter is introduced by the change of variable equations. The additional parameter for n = 1 is H(0) and for n = 2 is DH(0). The introduction of the parameter for each case requires the specification of four boundary conditions at the axis instead of the usual three. This conclusion was reached by Gawain [Ref. 13] and is analyzed in detail in his paper. There is one other peculiarity about the boundary conditions at the axis for n = 1 that should be mentioned. The axis boundary conditions in equation (3-18b) contain a pair of equations that involve the complex perturbation frequency γ, which represents an eigenvalue of the solution for the equations. The presence of γ in the axis boundary condition for n = 1 is an important factor in formulating a solution for the vorticity transport equations and is included in the eigensystem of the finite difference equations. The details of the implementation are discussed later.

The solution of the problem for predicting the onset of flow instabilities in circular pipes is now at hand. The original vorticity transport equations (2-40) are transformed to the functions P(r) and Q(r) and their respective derivatives by the appropriate change of variables depending on the specific angular wave number being investigated. After imposing the rigorously deduced boundary conditions, a solution can be determined in the perturbation quantities.

IV. PERTURBATION VELOCITY

The solution of the transformed vorticity transport equations developed in the previous chapter can now be expressed in terms of the perturbation quantities P(r) and Q(r). The original functions G(r) and H(r) are actually components of the vector potential function W defined in equation (2-28),

$$\vec{W} = [F(r)\,\vec{e}_x + G(r)\,\vec{e}_r + H(r)\,\vec{e}_\theta]\,e^{x} \qquad (2\text{-}28)$$

The axial component F(r) was previously set to zero since one function could be arbitrarily specified. From equation (2-27) the perturbation velocity vector was expressed as,

$$\vec{v} = \nabla \times \vec{W} \qquad (2\text{-}27)$$

Therefore the functions G(r) and H(r) represent the behavior of the perturbation velocity in the flow field. Referring back to the definition of the curl operation in equation (2-39), the relation for $\nabla \times \vec{W}$ has already been set up. The resulting expression for the perturbation velocity is given by,

$$\vec{v} = u\,\vec{e}_x + v\,\vec{e}_r + w\,\vec{e}_\theta \qquad (4\text{-}1)$$

The components of the perturbation velocity are,

$$u = DH(r) + \frac{H(r)}{r} - \frac{in}{r} G(r) \qquad (4\text{-}2)$$

$$v = -i\alpha H(r) \qquad (4\text{-}3)$$

$$w = i\alpha G(r) \qquad (4\text{-}4)$$

Digressing for a moment, it can be readily seen that since the perturbation velocity necessarily vanishes at the pipe wall, r = 1, the boundary conditions at the wall in equation (3-1) can be deduced.

Recall that the functions P(r) and Q(r) appear as the result of the change of variables for each specific angular wave number contained in subset (a) of equations (3-17) through (3-23). Analysis of the axial component of the perturbation velocity u becomes useful in interpreting the behavior of the functions P(r) and Q(r) along the radius of the pipe. Equations for the axial perturbation velocity in terms of P(r) and Q(r) are derived from equations (4-2) through (4-4) with the change of variable applied for each angular wave number being investigated and appear as follows,

$$n = 0 : \quad u = 2\,Q + r\,DQ \qquad (4\text{-}5)$$

$$n = 1 : \quad u = 3\,r\,Q + r^2 DQ - i\,r\,P \qquad (4\text{-}6)$$

$$n = 6 : \quad u = 4\,r^2\,Q + r^3 DQ - i\,6\,r^3\,P \qquad (4\text{-}7)$$

The axial perturbation velocity is computed in part VII of the main investigative program for each angular wave number. To prepare the perturbation velocity u for plotting versus pipe radius, it is first

normalized. Since this velocity is in general a complex quantity, the perturbation velocity with the largest magnitude is determined by,

$$|u_c|_{MAX} = (u_R^2 + u_I^2)^{\frac{1}{2}} \tag{4-8}$$

The normalizing velocity constant will then become u_c and the maximum normalized velocity will become,

$$\frac{u}{u_c} = 1 + i(0) \tag{4-9}$$

The remaining normalized velocities are found by dividing through by u_c. The normalized perturbation velocity is then plotted versus the normalized pipe radius for various axial wave numbers and Reynolds numbers at the three angular wave numbers investigated. The plots and results are discussed in detail in the chapter on Results.

V. NUMERICAL METHODS

A. GENERAL METHODS USED

In general, the vorticity transport equations are a pair of coupled homogeneous differential equations which are second order in the perturbation quantity G(r) and fourth order in the perturbation quantity H(r). A change of variables is introduced so that the number of boundary conditions at the axis is reduced to three for each angular wave number investigated, except for n = 1. Recall for this case that there are four boundary conditions imposed at the axis because of the introduction of the additional parameter H(0). By applying the change of variables to the original pair of equations, the vorticity transport equations are transformed into an equivalent pair of coupled homogeneous differential equations that are second order in P(r) and fourth order in Q(r). The solution of the vorticity transport equations will now be in terms of the perturbation quantities P(r) and Q(r) which are also referred to as eigenfunctions, and γ which is an eigenvalue. The axial perturbation velocity is expressed in terms of P(r) and Q(r) as derived previously and is examined to determine the behavior of the perturbation quantities. The resulting eigenvalue γ is of primary interest since the magnitude and sign of the real part of γ determines the relative stability or instability of the pipe flow problem.

A numerical solution of the transformed vorticity transport equations is now sought. The convenient matrix format of the equations is given below.

$$[M_4'] \begin{Bmatrix} D^4P \\ D^4Q \end{Bmatrix} + [M_3'] \begin{Bmatrix} D^3P \\ D^3Q \end{Bmatrix} + [M_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [M_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [M_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} =$$

$$\gamma\, [N_2'] \begin{Bmatrix} D^2P \\ D^2Q \end{Bmatrix} + [N_1'] \begin{Bmatrix} DP \\ DQ \end{Bmatrix} + [N_0'] \begin{Bmatrix} P \\ Q \end{Bmatrix} \qquad (5\text{-}1)$$

The details of the transformed coefficient matrices for n = 0, 1, and 6 are given in Appendix B. It is convenient to approximate the derivatives of P(r) and Q(r) by a finite number of discrete unknowns along a radial mesh consisting of N interior points. It is not possible to compute the continuous values of the functions and their derivatives along the pipe radius using a digital computer. Therefore, discrete finite difference techniques are used. For the problem at hand, the normalized nondimensional radius of the pipe, which becomes the independent variable, is divided into a computational mesh. The standard method of representing this mesh is with uniformly spaced stations over the interval $0 < r < 1$, Figure 5-1.

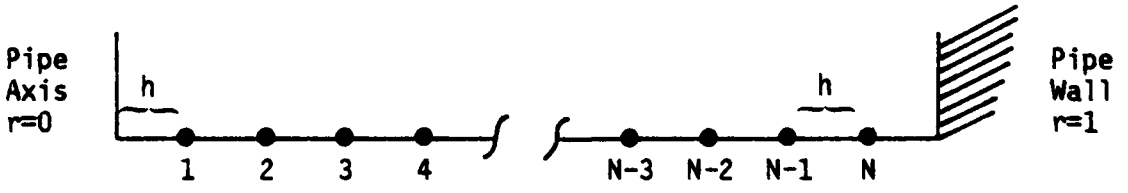


Figure 5-1. Radial Mesh, Standard Method

The vorticity transport equation coefficients are then evaluated at each station of the radial mesh resulting in N uniformily spaced discrete values with N + 1 equal intervals and N + 2 total points, including r = 0 and r = 1. The spacing between the stations is denoted by h = 1/(N + 1).

A refinement of the method for generating a computational mesh is referred to as the half station method and is used in this numerical analysis. All the radial stations of Figure 5-1 are moved to mid-interval or one half station (h/2) toward the axis with an additional station appearing at one half station from the pipe wall, as shown in Figure 5-2.

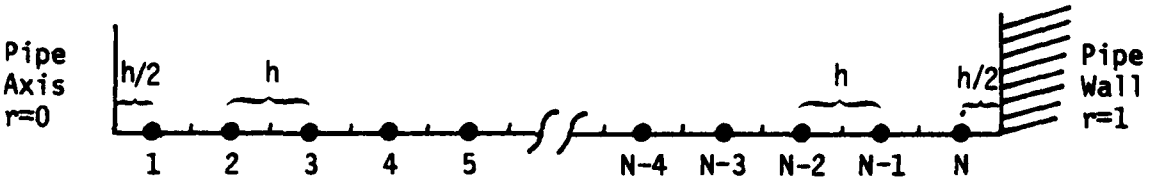


Figure 5-2. Radial Mesh, Half Station Method

Therefore for N interior stations, the method results in N intervals and N + 2 total points including the boundaries. The first station, r_1, is a distance h/2 from the axis and the Nth station, r_N, is a distance h/2 from the wall. The general expression for the ith station, corresponding to the ith increment of the radius, is given by,

$$r_i = h\,(i - \tfrac{1}{2})\ ;\ i = 1, 2, 3....N \qquad (5\text{-}2)$$

where h = 1/N.

An advantage of the half station method over the standard method is that there is better resolution at the stations near the boundaries where instabilities are thought to originate and where boundary layer effects are a factor. Therefore, the behavior of the perturbation quantities can be examined more closely. Additionally, better approximations for the derivatives of the functions P(r) and Q(r) can be realized with a smaller incremental distance to the points immediately adjacent to the boundaries. Even better resolution at the boundaries can be achieved by using a nonuniform computational mesh. Arnold [Ref. 11] introduced a change of independent variable by means of hyperbolic functions which included a mesh offset parameter so that he could control the concentration of mesh points at either the axis or wall. The method would certainly be useful in reducing computational time by reducing the total number of mesh points required for a satisfactory solution of the governing equations. However, the full implications of the nonuniform mesh with respect to α and γ dependence on the mesh offset parameter have not been fully explored. While this method is not used here, it may well prove to be useful in follow-on numerical investigations of the pipe flow problem.

The solution for the vorticity transport equations in terms of the perturbation quantities P(r) and Q(r) are now represented by a finite number of discrete unknowns. The governing differential equations are rewritten in finite difference form for N stations along the mesh resulting in a finite number of unknowns namely, P_1, P_2, P_3....P_N and Q_1, Q_2, Q_3....Q_N. In general, this produces 2N simultaneous equations in 2N

unknowns. The details of the derivation of the finite difference equations are explained shortly. Substituting the finite difference equations into the vorticity transport equation matrix format of equation (5-1) will result in N pairs of coupled simultaneous equations. These equations are regrouped and subsequently solved in the following eigenvalue problem format,

$$[A]\ \{X\} = \gamma\ [B]\ \{X\} \tag{5-3}$$

The column vector {X} represents the rearranged unknowns as

$$\{X\} = \begin{Bmatrix} P_1 \\ P_2 \\ \vdots \\ P_i \\ \vdots \\ P_N \\ Q_1 \\ Q_2 \\ \vdots \\ Q_i \\ \vdots \\ Q_N \end{Bmatrix} \tag{5-4}$$

The symbol γ denotes the characteristic value or eigenvalue of the solution. Details of the composition of the [A] and [B] matrices are discussed for each specific angular wave number investigated.

B. FINITE DIFFERENCE EQUATIONS

The use of finite differencing techniques is the basis for solving the governing vorticity transport equations in the eigensystem equation (5-3). It is required that the governing equations must be satisfied at each of the interior stations i = 1, 2, 3....N, therefore producing 2N equations in 2N unknowns.

Except for stations adjacent to the boundaries, the derivatives of the perturbation quantities in the governing equations at the interior stations are usually approximated by standard central finite difference equations. These equations were taken from Table 7.27 of Ketter and Prawel [Ref. 14] and are summarized below,

$$DF_i = \frac{1}{12h}\left\{F_{i-2} - 8F_{i-1} + 8F_{i+1} - F_{i+2}\right\} + \frac{1}{30}\,h^4D^5F_i \qquad (5\text{-}5)$$

$$D^2F_i = \frac{1}{12h^2}\left\{- F_{i-2} + 16F_{i-1} - 30F_i + 16F_{i+1} - F_{i+2}\right\} + \frac{1}{90}\,h^4D^6F_i \qquad (5\text{-}6)$$

$$D^3F_i = \frac{1}{8h^3}\left\{F_{i-3} - 8F_{i-2} + 13F_{i-1} - 13F_{i+1} + 8F_{i+2} - F_{i+3}\right\} + \frac{7}{120}\,h^4D^7F_i \qquad (5\text{-}7)$$

$$D^4F_i = \frac{1}{6h^4}\left\{-F_{i-3} + 12F_{i-2} - 39F_{i-1} + 56F_i - 39F_{i+1} + 12F_{i+2} - F_{i+3}\right\} + \frac{7}{240}\,h^4D^8F_i \qquad (5\text{-}8)$$

where F is an arbitrary function.

As can be seen from the equations above, the derivative of a function at a particular station can be approximated by a linear combination of the discrete unknown functions about a central point. The term outside the brackets in each equation represents the truncation error of the corresponding finite difference approximation. Notice that the truncation error of equations (5-5) through (5-8) are all of order h^4 (Oh^4). It was hypothesized and shown by Gawain and Ball [Ref. 15] that finite difference approximations of consistent order truncation error reduced the overall error of the numerical solution in a typical problem by a factor of five for truncation error Oh^2. For the present numerical analysis, a computational mesh of N = 50 was used for all cases. Now with the interval h = 1/N = 0.02, examination of the consistent fourth order truncation error term will yield an error of the order of $5x10^{-9}$ for all four derivatives approximated, since the magnitude of the normalized functions and derivatives are of order one. For an interval size of h = 0.02, a consistent truncation error Oh^4 should significantly reduce the overall error of the solution.

Special problems arise in the derivation of the finite difference equations at stations near the boundaries. Several approaches can be taken to approximate the derivatives at these stations, which include backward, forward and central finite differencing techniques. Unfortunately, the central method previously discussed will involve imaginary stations outside the boundaries for stations close to the axis and wall, i.e., stations i = -1, -2, -3, and N+1, N+2, N+3 are involved in the fourth derivative approximation at station i = 1 and i = N, respectively. Unlike the boundary conditions for many problems well suited for this

type of numerical analysis, the vorticity transport equation boundary conditions for the pipe flow problem are very complicated by comparison. The boundary conditions for each angular wave number are different; some involve additional parameters while others contain the eigenvalue γ. In other words, the conditions at the imaginary stations for conventional problems can normally be expressed in terms of conditions at the real stations in the computational mesh. It is therefore impractical to use the central method for the pipe flow problem because of the complicated nature of the boundary conditions.

The method of forward differencing is used near the pipe axis and backward differencing is used near the wall. These two differencing methods will be referred to, collectively, as noncentral finite differencing techniques. It will be shown later that the two methods are essentially equivalent. A rigorous derivation of the noncentral finite difference equations is based on the Taylor series (power series) expansion of the function at a point. Since the governing equations contain the fourth order derivative of Q(r), the derivation for this case is examined in detail here.

The Taylor series approximation of $Q_i(r)$ at station i = 1 appears as equation (5-9).

$$Q_1 = Q(0) + DQ(0)\,\frac{h}{2} + D^2Q(0)\,\frac{(\frac{h}{2})^2}{2!} + D^3Q(0)\,\frac{(\frac{h}{2})^3}{3!} + D^4Q(0)\,\frac{(\frac{h}{2})^4}{4!}$$

$$+ D^5Q(0)\,\frac{(\frac{h}{2})^5}{5!} + D^6Q(0)\,\frac{(\frac{h}{2})^6}{6!} + D^7Q(0)\,\frac{(\frac{h}{2})^7}{7!} + \ldots . \qquad (5\text{-}9)$$

The approximation for Q_i at stations i = 1, 2, 3, 4, 5, 6 are expressed in convenient matrix format below,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} 1 & \frac{1}{2} & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} & \frac{(\frac{1}{2})^6}{6!} & \frac{(\frac{1}{2})^7}{7!} \\ 1 & \frac{3}{2} & \frac{(\frac{3}{2})^2}{2!} & \frac{(\frac{3}{2})^3}{3!} & \frac{(\frac{3}{2})^4}{4!} & \frac{(\frac{3}{2})^5}{5!} & \cdot & \cdot \\ 1 & \frac{5}{2} & \frac{(\frac{5}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{7}{2} & \frac{(\frac{7}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{9}{2} & \frac{(\frac{9}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \\ 1 & \frac{11}{2} & \frac{(\frac{11}{2})^2}{2!} & \cdot & \cdot & \cdot & \cdot & \cdot \end{bmatrix}_{6\times 8} \begin{Bmatrix} Q(0) \\ hDQ(0) \\ h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + \{E_1\}\, h^8D^8Q(0) \qquad (5\text{-}10)$$

where the dots represent matrix elements that can be easily deduced.

Since the fourth order derivative of Q(r) is present in the vorticity transport equations, there will be two boundary conditions specified at the axis and two at the wall. It can be shown that two columns of the matrix in equation (5-10) can be eliminated since two of the functions in the column vector are either zero or can be expressed in terms of additional parameters. The matrix, which will be denoted as [AA], now becomes a square 6 X 6 matrix. Notice that the column vector $\{E_1\}$ represents the last nonvanishing term in the Taylor series. The approximation for the first derivative of $Q_i(r)$ at station i = 1 is given by,

$$DQ_1 = DQ(0) + D^2Q(0)\,\frac{h}{2} + D^3Q(0)\,\frac{(\frac{h}{2})^2}{2!} + D^4Q(0)\,\frac{(\frac{h}{2})^3}{3!} + D^5Q(0)\,\frac{(\frac{h}{2})^4}{4!}$$

$$+ D^6Q(0)\,\frac{(\frac{h}{2})^5}{5!} + D^7Q(0)\,\frac{(\frac{h}{2})^6}{6!} + \ldots\ldots \qquad (5\text{-}11)$$

The Taylor series expansion for the second, third, and fourth derivative can be found by successive deferentiation of equation (5-11). The first four derivatives of Q_i at station i = 1 can be expressed in matrix format as,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \underset{4\times 8}{\begin{bmatrix} 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} & \frac{(\frac{1}{2})^6}{6!} \\ 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} & \frac{(\frac{1}{2})^5}{5!} \\ 0 & 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} & \frac{(\frac{1}{2})^4}{4!} \\ 0 & 0 & 0 & 0 & 1 & (\frac{1}{2}) & \frac{(\frac{1}{2})^2}{2!} & \frac{(\frac{1}{2})^3}{3!} \end{bmatrix}} \begin{Bmatrix} Q(0) \\ hDQ(0) \\ h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + \{E_2\}\, h^8D^8Q(0) \qquad (5\text{-}12)$$

It can be shown here as well that two columns of the matrix in equation (5-12) can be eliminated so that the matrix denoted [CC] is now a 4 X 6 matrix. The column vector $\{E_2\}$ also represents the last non-vanishing term in the Taylor series. It can be seen that the approximations for the derivatives of Q_i at stations i = 2 and i = 3 can be determined by replacing (1/2) in the [CC] matrix with (3/2) and (5/2), respectively.

To show how the noncentral finite difference approximations near the axis are formulated, the case for angular wave number $n \geq 6$ is examined. The boundary conditions at the axis for the function Q are,

$$Q(0) = 0 \qquad \text{and} \qquad DQ(0) = 0 \tag{5-13}$$

Applying these conditions to equations (5-10) and (5-12) will eliminate the first two columns of the [AA] and [CC] matrices since the first two terms of the column vectors vanish. The truncation error terms $\{E_1\}$ and $\{E_2\}$ are very small quantities and are neglected at this point. In shorthand notation, equation (5-10) now becomes,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \underset{6\times6}{[AA]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} \tag{5-14}$$

and equation (5-12) becomes,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \underset{4\times6}{[CC]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} \tag{5-15}$$

Equation (5-14) is multiplied through $[AA]^{-1}$ and becomes,

$$\begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} = \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} \tag{5-16}$$

and is then substituted into equation (5-15) giving,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \underset{4\times6}{[CC]} \quad \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} \tag{5-17}$$

Now, the fourth derivative of Q_1 at station i = 1 is approximated in terms of the discrete functions of Q at the interior mesh points with truncation error Oh^4. The values on the bottom row of the resulting 4 X 6 matrix after multiplication in equation (5-17) are the coefficients of the column vector Q_1 through Q_6 for D^4Q_1. In order to have consistent truncation error Oh^4, the last row and last column of the [AA] and [CC] matrices are deleted for the third derivative of Q. Matrix [AA] becomes a 5 X 5 and is now inverted and matrix [CC] becomes a 3 X 5. The values on the bottom row of the resulting 3 X 5 matrix are

the coefficients of Q_1 through Q_8 which approximates D^3Q_1. The rows and columns of matrices [AA] and [CC] are successively reduced to determine the coefficients for D^2Q_1 in terms of Q_1 through Q_4, and DQ_1 in terms of Q_1 through Q_3 so that consistent truncation error Oh^4 is preserved throughout. The noncentral finite difference equation for D^4Q_1 will now appear as,

$$D^4Q_1 = \frac{1}{h^4}\,(A_1Q_1 + A_2Q_2 + A_3Q_3 + A_4Q_4 + A_5Q_5 + A_6Q_6) + Oh^4 \qquad (5\text{-}18)$$

where A_1 through A_6 are the coefficients taken from the bottom row of the resulting matrix in equation (5-17)

In a similar manner, by imposing the required boundary conditions, this process can be carried out to determine the noncentral finite difference equations for the first two derivatives of $P(r)$ at several points near the boundaries as required.

It is important to note here that the noncentral finite difference approximations near the wall can be formulated by using the same techniques as those used near the axis. Once again taking the case for angular wave number $n \geq 6$, the boundary conditions at the pipe wall for the function Q are,

$$Q(1) = 0 \qquad \text{and} \qquad DQ(1) = 0 \qquad (5\text{-}19)$$

Rather than set up complete new [AA] and [CC] matrices at different values of r in the Taylor series expansion near the wall, assume that the boundary conditions in equation (5-19) are imposed at the axis,

r = 0. Notice that the boundary conditions in equation (5-19) are now identical to the boundary conditions in equation (5-13). The method for computing the coefficients of the noncentral finite difference approximations at the wall can be carried out as discussed previously and, in this case, have already been computed. It can be shown that the A_i coefficients of equation (5-18) are applicable to D^4Q_N at the wall but will appear in reverse order. The noncentral finite difference equation for D^4Q_N is given by,

$$D^4Q_N = \frac{1}{h^4}\left(A_6Q_{N-5} + A_5Q_{N-4} + A_4Q_{N-3} + A_3Q_{N-2} + A_2Q_{N-1} + A_1Q_N\right) + Oh^4 \tag{5-20}$$

The method for determining the coefficients for the remaining derivatives of Q and P near the wall proceed as though the boundary conditions were imposed at the axis; the coefficients are computed, their order reversed and coefficients of the odd order derivatives undergo a sign change. Order reversal and sign changes are accomplished in part II of the main investigative program when the precomputed noncentral finite difference coefficients are read in as data.

As can be seen from the central finite difference approximations of equations (5-5) through (5-8), the third and fourth derivatives of a function require three points either side of the central point being approximated. The first and second derivatives require only two points either side of the central point. Therefore the central finite difference approximations are applicable only to interior stations i = 4 through N-3 for D^4Q_i and D^3Q_i, and stations i = 3 through N-2 for D^2Q_i,

DQ_i, D^2P_i, and DP_i. This leaves either two or three points immediately adjacent to the boundaries to be approximated by the noncentral finite difference equations just developed.

Since the central finite difference approximations are applicable to the above mentioned interior mesh points and are valid for all the derivatives of P(r) and Q(r) as well, the coefficients are computed only once in part I of the main investigative program in double precision format. The coefficients of the noncentral finite difference approximations, which are dependent on the boundary conditions for each angular wave number, are computed in an independent computer program . The resulting noncentral coefficients for the derivatives of P(r) and Q(r) at mesh points near the boundaries are also computed in double precision format and are read into part II of the main investigative program as data.

C. SPECIFIC METHODS FOR n = 0, 1, AND 6

The vorticity transport equations have now been expanded into a set of discrete simultaneous equations with respect to a half station computational mesh of N points (stations). The derivatives of the governing equations were approximated by central and noncentral finite difference coefficients, with consistent fourth order truncation error, in terms of the discrete unknown functions of the system. The convenient matrix format of the eigensystem in equation (5-3) can now be solved on a digital computer using a specialized Fortran subroutine, which will be discussed later. It is important to understand the composition of the [A] and [B] matrices and the column vector of unknowns

{X}. The problem setup is examined in detail for each angular wave number investigated. In the following discussions, the vorticity transport equations are referred to in the transformed state after the change of variables to P(r) and Q(r) have been made. The boundary conditions in equations (3-17), (3-18), and (3-23), and the coefficients in Appendix B also apply.

For the case n = 0, the pipe flow problem becomes greatly simplified in that the equations become uncoupled and only the solution for Q(r) is investigated for reasons explained earlier. The governing equation in terms of Q(r) is represented by equation (2-38). At the outset, it was determined that a mesh size of N = 50 was sufficient to produce a satisfactory numerical solution of the problem. It was necessary to determine when the solution was relatively independent of mesh size. Preliminary trial computations were performed for various mesh sizes, N = 25 to 100. From N = 45 to 100, the solution of the equation for n = 0 did not vary appreciably. Additionally, the solution was relatively independent of Reynolds numbers below 15,000 for mesh sizes of about 50. In order to reduce computation time and the eigensystem matrix size, a mesh of N = 50 was used in this analysis.

In general for angular wave number n = 0, the [A] and [B] matrices are 50 X 50 and appear as banded diagonal matrices. For a consistent order truncation error Oh^4 used here, the derivative of highest order requires the greatest band width. From the governing equation in Q, it can be seen that the highest order derivative in [A] is D^4Q and in [B] is D^2Q. At stations near the boundaries, the noncentral finite difference coefficients require a band width of six in matrix [A] and four in

matrix [B]. At interior stations, the central finite difference coefficients require a band width of seven for [A] and five for [B]. Therefore the [A] and [B] matrices of equation (5-3) contain nonzero elements as shown in Figures 5-3 and 5-4. The X elements contain non-central finite difference coefficients and the 0 elements contain central difference coefficients. The X and 0 symbols denote the station at which the approximation is made. The composition of the column vector {X} can also be seen in Figures 5-3 and 5-4.

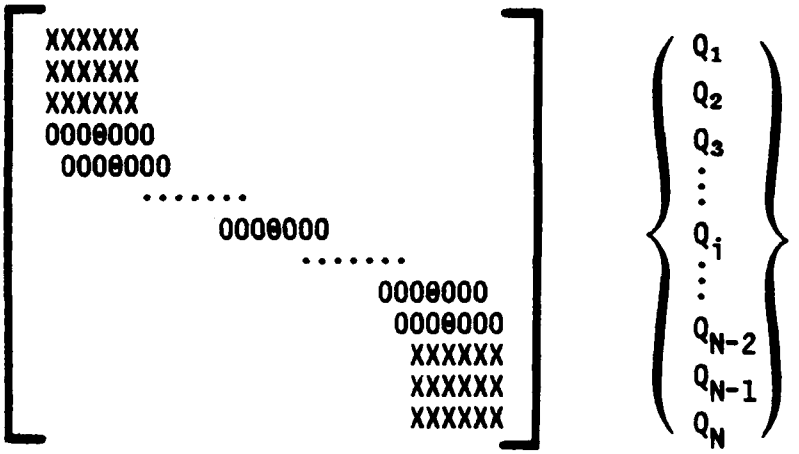


Figure 5-3. [A] Matrix for n = 0

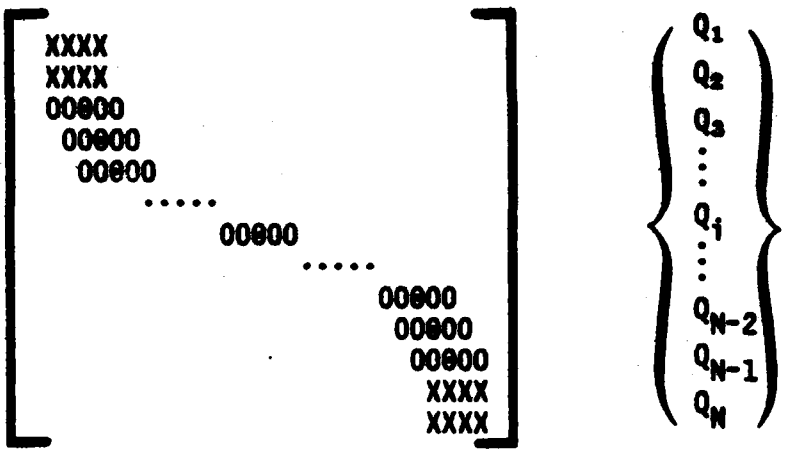


Figure 5-4. [B] Matrix for n = 0

The actual values of the elements of the [A] and [B] matrices are computed in part IV of the main investigative program. The calculations are rather lengthy and tedious and are accomplished in iterative loops in part IV. To save computation time, calculations are made only for the nonzero elements of the matrices. Recall that the governing equation in Q(r) for n = 0 was given in equation (2-43). In discrete form for the ith station, equation (2-43) appears as,

$$M'_{4_i} D^4Q_i + M'_{3_i} D^3Q_i + M'_{2_i} D^2Q_i + M'_{1_i} DQ_i + M'_{0_i} Q_i =$$

$$\gamma(N'_{2_i} D^2Q_i + N'_{1_i} DQ_i + N'_{0_i} Q_i) \qquad (5\text{-}21)$$

where the primed quantities are element (2,2) of the corresponding transformed vorticity transport equation matrices given in Appendix B, Part A. At station i = 1, the top row of the [A] matrix will have six elements and the top row of [B] will have four elements.

For example, the matrix elements for the first row of [A] can be found as follows, where the a_i coefficients are analogous to the coefficients A_i of equation (5-18) and are equal to $a_i = A_i/h^4$. Similarly $b_i = B_i/h^3$, $c_i = C_i/h^2$, and $d_i = D_i/h$, where b_i, c_i, and d_i are the noncentral finite difference coefficients of the remaining three derivatives.

$$A_{1,1}\{Q_1\} = (M'_{4_1} a_1 + M'_{3_1} b_1 + M'_{2_1} c_1 + M'_{1_1} d_1 + M'_{0_1})\{Q_1\} \quad (5\text{-}22)$$

$$A_{1,2}\{Q_2\} = (M'_{4_1} a_2 + M'_{3_1} b_2 + M'_{2_1} c_2 + M'_{1_1} d_2)\{Q_2\} \quad (5\text{-}23)$$

$$A_{1,3}\{Q_3\} = (M'_{4_1} a_3 + M'_{3_1} b_3 + M'_{2_1} c_3 + M'_{1_1} d_3)\{Q_3\} \quad (5\text{-}24)$$

$$A_{1,4}\{Q_4\} = (M'_{4_1} a_4 + M'_{3_1} b_4 + M'_{2_1} c_4)\{Q_4\} \quad (5\text{-}25)$$

$$A_{1,5}\{Q_5\} = (M'_{4_1} a_5 + M'_{3_1} b_5)\{Q_5\} \quad (5\text{-}26)$$

$$A_{1,6}\{Q_6\} = (M'_{4_1} a_6)\{Q_6\} \quad (5\text{-}27)$$

where $A_{1,1}$ is the matrix element in the first row and the first column of [A], and so forth. This procedure is carried out for N equations along the computational mesh until all the nonzero matrix elements of [A] and [B] are computed.

Derivation of the noncentral finite difference coefficients for other angular wave numbers is fairly straightforward when accomplished by the methods outlined in the previous section. The boundary conditions for Q at the axis for n = 0 are,

$$DQ(0) = 0 \qquad \text{and} \qquad D^3Q(0) = 0 \tag{5-28}$$

After imposing these conditions, the second and fourth columns of the matrices in equations (5-10) and (5-12) can be eliminated since the corresponding terms in the column vectors vanish. The coefficients are then computed as described before in an independent computer program and then read into the main investigative program as data. Notice that the bondary conditions for the function Q at the wall are the same for $n = 0$ and $n \geq 6$. Therefore, the noncentral finite difference coefficients need to be computed only once.

The case for angular wave number $n \geq 6$ will be examined next since it more nearly represents the general case. Since the vorticity transport equations remain coupled for $n \geq 6$ and no additional parameters are introduced, a system of 2N equations in 2N unknowns will result. For a mesh size of 50, the [A] and [B] matrices become 100 X 100 and are solved in the form of equation (5-3). The first N equations of the system are represented by the discrete form of equation (2-37), which after transformation by the appropriate change of variables corresponds to the top equation of (5-1). The second N set of equations are represented by the discrete form of equation (2-38), which after transformation by the appropriate change of variables corresponds to the bottom equation of (5-1). As a matter of convenience for programming, the [A] and [B] matrices are partitioned into four 50 X 50 quadrants, which are individually banded diagonal matrices. The properties of the [A] and [B] matrices for $n \geq 6$ are summarized in Table 5-1.

Table 5-1. Properties of the [A] and [B] Matrices for $n \geq 6$

MATRIX	QUADRANT	HIGHEST ORDER DERIVATIVE PRESENT	NONCENTRAL COEFFICIENTS REQUIRED	*	CENTRAL COEFFICIENTS REQUIRED
A	I	D^2P	5	2	5
A	II	D^3Q	5	3	7
A	III	D^2P	5	2	5
A	IV	D^4Q	6	3	7
B	I	P	-	-	1
B	II	DQ	3	2	5
B	III	P	-	-	1
B	IV	D^2Q	4	2	5

* Number of stations near the boundaries where the noncentral finite difference approximations apply.

It can be deduced from the data given in Table 5-1 that the [A] and [B] matrices will contain nonzero elements as shown in Figures 5-5 and 5-6. The column vector {X} of unknowns is also given.

The method for determining the noncentral finite difference coefficients for $n \geq 6$ was discussed in a previous example.

The case for $n = 1$ is unique in that an additional parameter H(0) is introduced by the change of variables and a pair of coupled boundary conditions at the axis in equation (3-18b) contain the eigenvalue γ. These peculiarities are significant factors in correctly setting up the problem. Recall that in the coupled governing equations (2-47) the additional parameter H(0) appears explicitly and two additional column

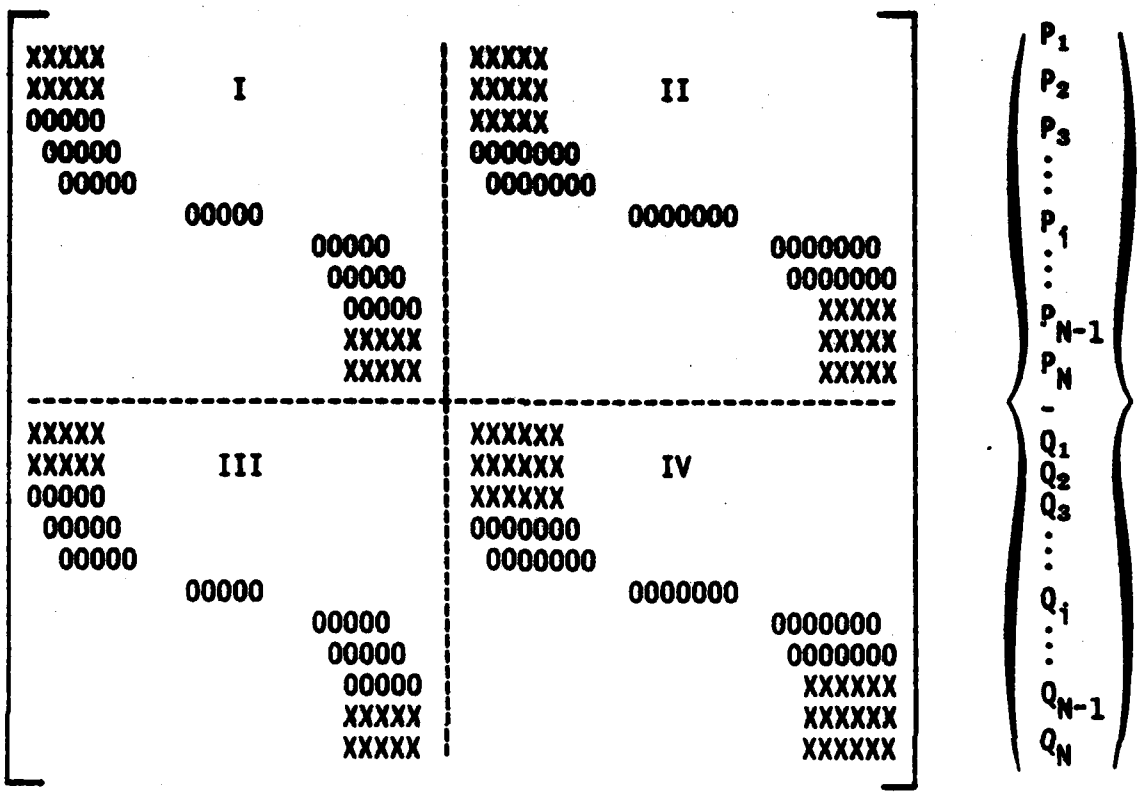


Figure 5-5. [A] Matrix for n ≥ 6

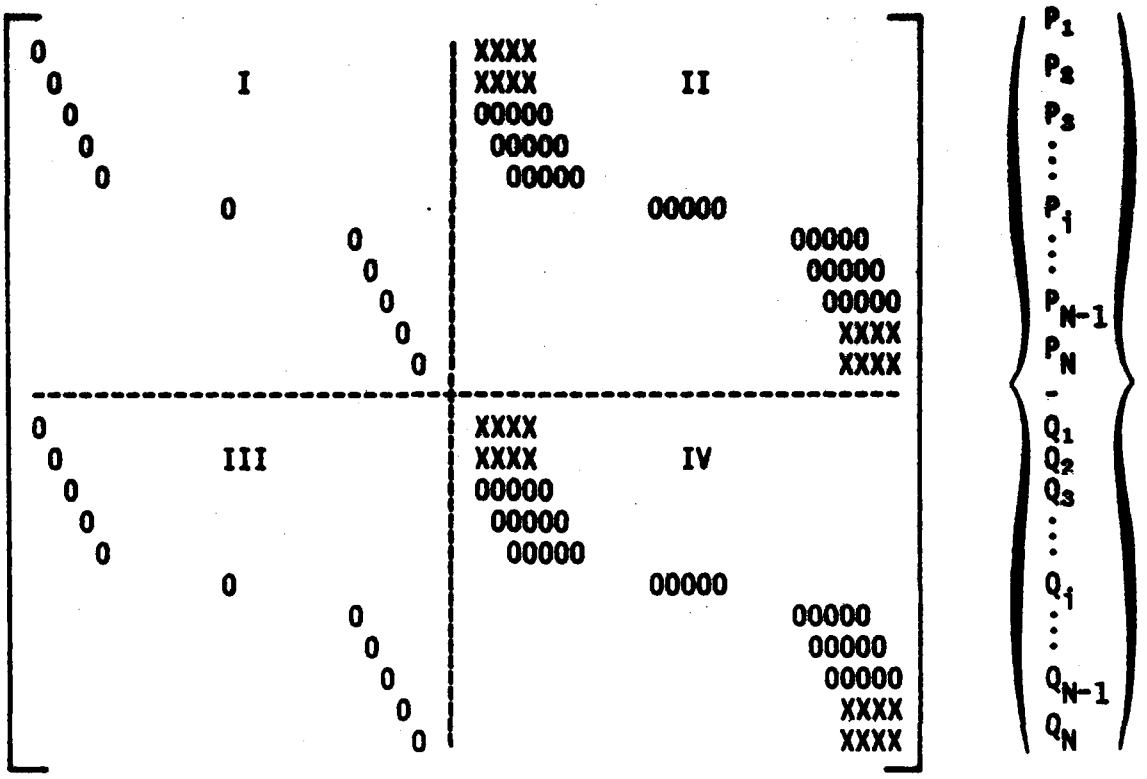


Figure 5-6. [B] Matrix for n ≥ 6

matrices are required in the system of equations. This was due to the change of variables introduced by equations (2-44). The appearance of the eigenvalue γ in the pair of boundary equations at the axis comes about from the rigorous method in which the boundary conditions were derived in Chapter III. The coupled axis boundary conditions in equation (3-18b) are repeated here,

$$-\frac{96}{Re}[C_7]\begin{Bmatrix} D^2P(0) \\ D^2Q(0) \end{Bmatrix} + 2i\alpha[C_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + i\alpha[C_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)$$

$$= \gamma\left(2[D_8]\begin{Bmatrix} P(0) \\ Q(0) \end{Bmatrix} + \alpha^2[D_9]\begin{Bmatrix} -i \\ 1 \end{Bmatrix} H(0)\right) \qquad (5\text{-}29)$$

where the coefficient matrices are contained in Appendix C. Notice that the parameter H(0) appears explicitly in these two equations as well. Additionally, the parameters $D^2P(0)$, $D^2Q(0)$, P(0), and Q(0) are also introduced. Obviously, this poses some problems since there are only two additional equations and five unknown parameters. H(0) can be used since it appears in the 2N discrete equations of the system. Q(0) is chosen as the second variable as a matter of convenience. The remaining three functions can be approximated in terms of discrete functions of P_i in the case for $D^2P(0)$ and P(0), and Q_i and Q(0) in the case of $D^2Q(0)$. By referring to the matrix form of the Taylor series expansion of a function at a point in equation (5-10), it can be seen how the approximations of these functions are made. For the present situation where

DP(0) = 0, eliminate the second column of the matrix and equation (5-10) will become,

$$\begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} = \underset{5\times5}{[AA]} \begin{Bmatrix} P(0) \\ h^2D^2P(0) \\ h^3D^3P(0) \\ h^4D^4P(0) \\ h^5D^5P(0) \end{Bmatrix} + Oh^6 \qquad (5\text{-}30)$$

Recall that the matrix in equation (5-30) is denoted [AA] as before. After multiplying each side of the equation by $[AA]^{-1}$, the result becomes,

$$\begin{Bmatrix} P(0) \\ h^2D^2P(0) \\ h^3D^3P(0) \\ h^4D^4P(0) \\ h^5D^5P(0) \end{Bmatrix} = \underset{5\times5}{[AA]^{-1}} \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \end{Bmatrix} - Oh^6 \qquad (5\text{-}31)$$

Now P(0) is approximated by the coefficients in the first row of $[AA]^{-1}$ and $D^2P(0)$ is approximated by the coefficients in the second row of $[AA]^{-1}$ divided by h^2, both expressed in terms of P_1 through P_5. Notice that the approximation for $D^2P(0)$ has truncation error Oh^4 and P(0) has truncation error Oh^6, which is acceptable in this analysis.

The approximation for $D^2Q(0)$ is similar except that Q(0) will appear explicitly in the approximations with DQ(0) = 0. Equation (5-10) now becomes,

$$\begin{Bmatrix} Q_1 \\ Q_2 \\ Q_3 \\ Q_4 \\ Q_5 \\ Q_6 \end{Bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} \{Q(0)\} + \underset{6\times6}{[AA]} \begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} + Oh^8 \qquad (5\text{-}32)$$

After rearranging, equation (5-32) is multiplied through by $[AA]^{-1}$ and becomes,

$$\begin{Bmatrix} h^2D^2Q(0) \\ h^3D^3Q(0) \\ h^4D^4Q(0) \\ h^5D^5Q(0) \\ h^6D^6Q(0) \\ h^7D^7Q(0) \end{Bmatrix} = \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 - Q(0) \\ Q_2 - Q(0) \\ Q_3 - Q(0) \\ Q_4 - Q(0) \\ Q_5 - Q(0) \\ Q_6 - Q(0) \end{Bmatrix} - Oh^8 \qquad (5\text{-}33)$$

The coefficients for $D^2Q(0)$ are taken from the first row of $[AA]^{-1}$ and divided by h^2. $D^2Q(0)$ is now expressed in terms of $Q(0)$ and Q_1 through Q_6 and will have a truncation error Oh^6.

The two additional boundary condition equations involving γ can now be solved in the eigensystem of equations with $H(0)$ and $Q(0)$ appearing explicitly as the two required unknowns. It is convenient to express the $[C_9]$ and $[D_9]$ matrices of equation (5-29) in slightly different form,

$$[C_9]\begin{Bmatrix}-i\\1\end{Bmatrix} H(0) \quad \text{as} \quad [C_9^*]\,\{H(0)\}$$

and (5-34)

$$[D_9]\begin{Bmatrix}-i\\1\end{Bmatrix} H(0) \quad \text{as} \quad [D_9^*]\,\{H(0)\}$$

where $[C_9^*]$ and $[D_9^*]$ become 2 X 1 column matrices,

$$[C_9^*] = \begin{bmatrix}\left(\frac{\alpha^3}{Re} - 4i + 2i\alpha^2\right)\\ \left(-2\alpha^2 + \frac{i\alpha^3}{Re}\right)\end{bmatrix} \quad \text{and} \quad [D_9^*] = \begin{bmatrix}-i\\ \\ 1\end{bmatrix} \tag{5-35}$$

Substituting equations (5-35) into equation (5-29) yields,

$$-\frac{96}{Re}[C_7]\begin{Bmatrix}D^2P(0)\\D^2Q(0)\end{Bmatrix} + 2i\alpha[C_8]\begin{Bmatrix}P(0)\\Q(0)\end{Bmatrix} + i\alpha[C_9^*]\,\{H(0)\}$$

(5-36)

$$= \gamma\left(2\,[D_8]\begin{Bmatrix}P(0)\\Q(0)\end{Bmatrix} + \alpha^2\,[D_9^*]\,\{H(0)\}\right)$$

The solution of the eigensystem of 2N + 2 equations in 2N + 2 unknowns is now accomplished in the form of equation (5-3). The appearance of the [A] and [B] matrices are identical to those for $n \geq 6$ in Figures 5-5 and 5-6 except for the addition of two columns to accommodate the parameters H(0) and Q(0), and two rows to accommodate the pair of boundary equations in (5-36). Nonzero elements are present in the [A] and [B] matrices for n = 1 as shown in Figures 5-7 and 5-8.

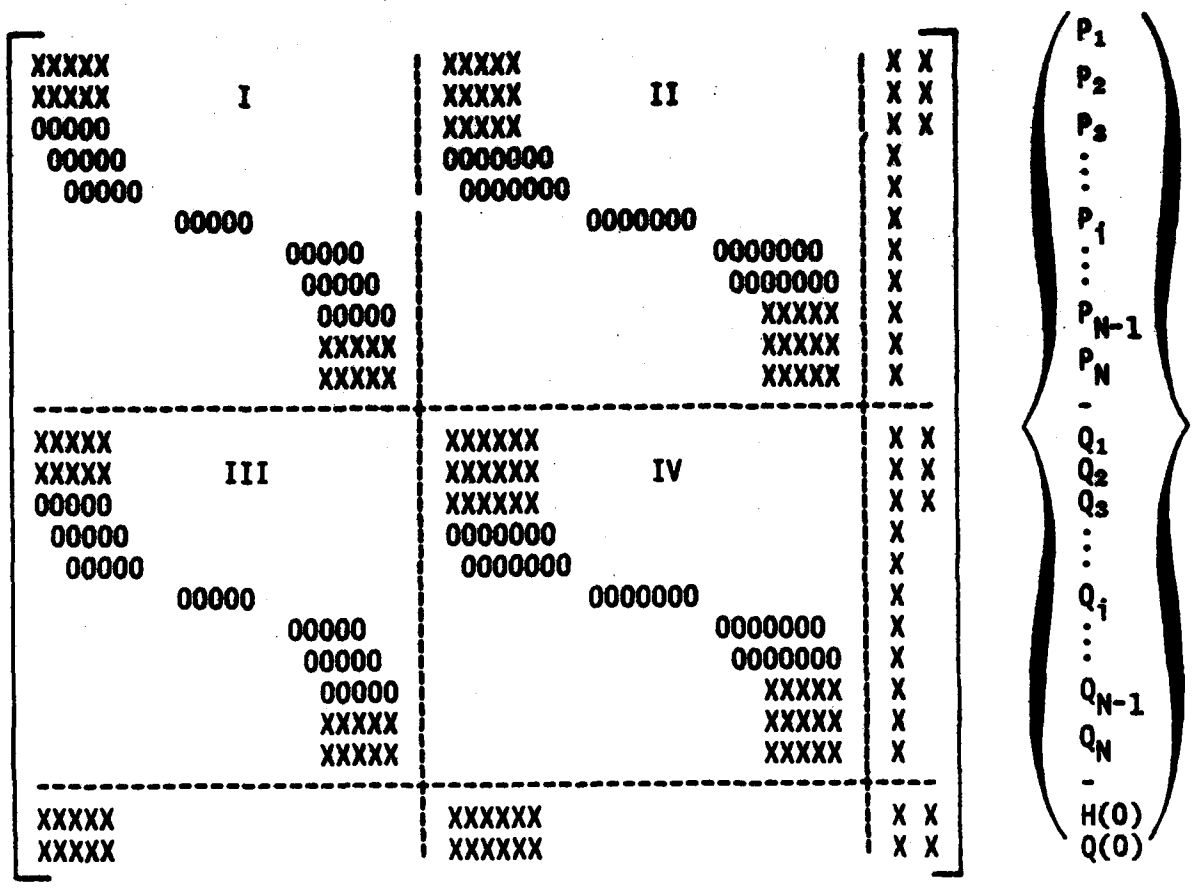


Figure 5-7. [A] Matrix for n = 1

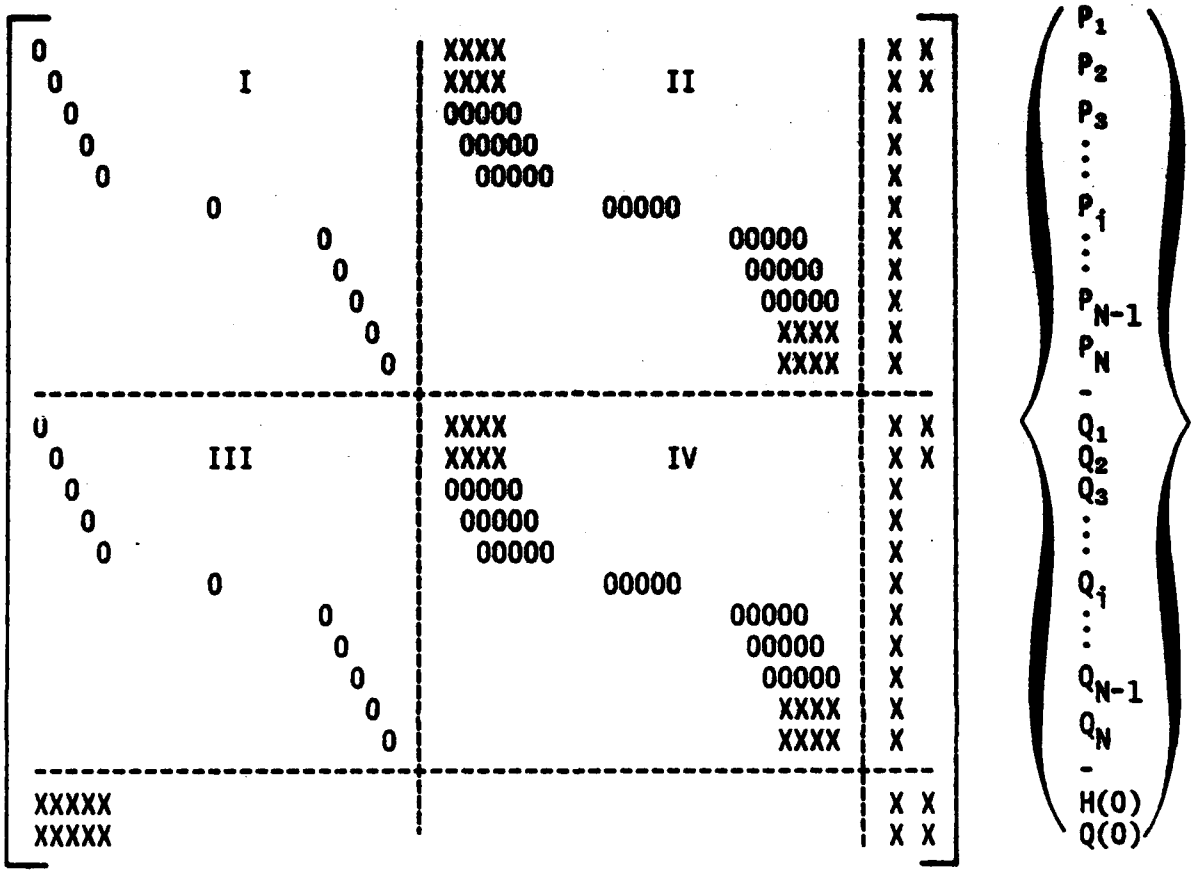


Figure 5-8. [B] Matrix for n = 1

Derivation of the noncentral finite difference coefficients for the derivatives of P and Q at the wall are straightforward as shown for the other angular wave numbers. The columns of the [AA] and [CC] matrices are eliminated based on the fact that parameters P(1), Q(1), and DQ(1) are all expressed in terms of H(0), as seen in equations (3-18c). By imposing the boundary conditions for Q at the wall,

$$Q(1) = - H(0) \quad \text{and} \quad DQ(1) = 2\,H(0) \tag{5-37}$$

equation (5-17) becomes,

$$\begin{Bmatrix} hDQ_1 \\ h^2D^2Q_1 \\ h^3D^3Q_1 \\ h^4D^4Q_1 \end{Bmatrix} = \begin{bmatrix} 2h \\ 0 \\ 0 \\ 0 \end{bmatrix} \{H(0)\} + \underset{4\times6}{[CC]}\ \underset{6\times6}{[AA]^{-1}} \begin{Bmatrix} Q_1 + (1-h)\,H(0) \\ Q_2 + (1-3h)\,H(0) \\ Q_3 + (1-5h)\,H(0) \\ Q_4 + (1-7h)\,H(0) \\ Q_5 + (1-9h)\,H(0) \\ Q_6 + (1-11h)\,H(0) \end{Bmatrix} \tag{5-38}$$

The coefficients in the resultant 4 X 6 matrix of equation (5-38) are in fact the same coefficients computed for $n \geq 6$ at the wall, and recall that they appear in reverse order when used at the wall. The details of the complete derivation of the noncentral finite difference coefficients near the boundaries is lengthy and not included here.

D. COMPUTER PROGRAM USEAGE

The solutions for the discrete transformed vorticity transport equations for the three angular wave numbers investigated are accomplished in the main investigative computer programs. Three separate programs were written rather than developing one program for the general case. The rationale for this is as follows: the changes of variables for each angular wave number $n \leq 6$ are different, the boundary conditions are different, the equations uncouple for $n = 0$, additional parameters are introduced for some, and $n = 1$ has a coupled boundary equation that contains γ that must be dealt with separately in the setup of the problem. It would be practical to write a general program for angular wave numbers $n \geq 6$, however.

For each specific angular wave number analyzed, the main computer program is divided into eight parts. Their functions are summarized below:

Part I — The central finite difference coefficient for the derivatives of P and Q are computed.

Part II — The noncentral finite difference coefficients for the derivatives of P and Q at the axis and wall are read in as data. Problem variables, Re and α, are read in as data.

Part III — Coefficients of the vorticity transport matrix equations are computed along the radial mesh.

Part IV — The [A] and [B] matrix elements are computed.

Part V — The eigenvalues and respective eigenvectors are computed in the subroutine EIGZC and the least stable eigenvalue is determined.

Part VI — Verification that the least stable eigenvalue and corresponding eigenvector satisfy the governing equations.

Part VII The normalized axial perturbation velocity is computed from the least stable eigenvalue and its eigenvector.

Part VIII The normalized axial perturbation velocity is plotted versus normalized pipe radius utilizing the Versatec plotter.

The numerical analysis of the pipe flow stability problem is centered around the subroutine EIGZC once the problem is properly set up. The eigensystem of equation (5-3) is solved by this subroutine which returns a set of N, 2N + 2, or 2N eigenvalues for n = 0, 1, or 6, respectively, and a corresponding set of normalized eigenvectors. Each eigenvalue and its eigenvector represent a solution of the governing equations. The primary objective of the main program and of this investigation is to determine the least stable eigenvalue which indicates flow characteristics. Recall that if γ_R is positive, the flow is unstable. For each value of α and Re chosen, the maximum algebraic value of γ_R is found, denoted as γ^*, to determine if instabilities exist in the flow field. The corresponding eigenvector of γ^*, denoted as $\{X^*\}$, is used to compute the incremental axial perturbation velocity along the radius. Plots of the normalized perturbation velocity versus normalized pipe radius have no significance in determining stability or instability of the flow but are indicators of the general nature and behavior of the flow field perturbations. Another independent program was written to plot Reynolds number stability contours using data taken from program results for n = 1 and 6. The subroutines used to make plots on Versatec plotter are PLOTG, standard Versatec plotting functions and TITLE1 to print titles and legends on the plots.

The numerical accuracy of the solution was determined by substituting γ^* and its eigenvector into the governing equations. For this purpose, the residual error vector $\{\varepsilon\}$ was computed by equation (5-39) and the magnitude of the error was examined.

$$\{\varepsilon\} = [A]\{X^*\} - \gamma^*[B]\{X^*\} \tag{5-39}$$

The Fortran subroutine EIGZC is contained in the International Mathematical and Statistical Library (IMSL) available in most large computer system libraries utilizing Fortran compilers. Specifically, the numerical analysis of the vorticity transport equations in the form of equation (5-3) was computed on the IBM 370/3033 (assigned processor) system with the Fortran H Extended compiler at the U.S. Naval Postgraduate School. All calculations were performed in full double precision format to improve the accuracy of the solution by minimizing truncation and round off errors inherently introduced by digital computations. The main investigative programs and other auxiliary programs were all written in Fortran programming language to be compatible with all versions of Fortran compilers. The computer program listings with applicable data sets are included after the appendices in this paper.

VI. RESULTS

A. RESULTS FOR n = 0

Results of the pipe flow stability problem are obtained from the solution of the linearized vorticity transport equations by the methods previously described. A generalized solution of the governing differential equations for fully developed, three-dimensional pipe flow has been accomplished for three angular wave numbers, n = 0, 1, and 6. The characteristics of the flow field perturbations for each angular wave number investigated are established by the parameters Re and α. The purpose of this numerical investigation was to determine flow stability at various Reynolds numbers and axial wave numbers for each angular wave number. Recall that the perturbation velocity vector $\vec{v}$ is defined in equation (2-27) as the curl of the velocity vector potential $\vec{W}$. This function is expressed in terms of G(r) and H(r) and a complex exponential form e^{X} in equations (2-28) and (2-29). After appropriate changes of variables were introduced, solutions were determined from the eigensystem of equations which consists of a set of eigenvalues and their corresponding normalized eigenvectors for fixed values of Re and α. The resulting perturbation quantities now appear in terms of the eigenfunctions P(r) and Q(r) and are represented in discrete form in the eigenvector {X} as P_i and Q_i, where i = 1, 2, 3.....N. As can be seen from equation (2-30), the real part of the complex eigenvalue γ_R will determine the growth or decay rate in time of the perturbation. Positive

values of γ_R indicate that the flow is unstable. The eigenvalue whose real part has the largest algebraic value, denoted as γ^*, is the least stable and is reported in these results to show stability trends.

Gawain [Ref. 13] reported the numerical results for angular wave number n = 0 for the solution of the governing equation in his paper. Those numerical results which incorporated a purely imaginary axial wave number and the advancements in the derivation of the boundary conditions at the pipe axis were duplicated in this investigation. They are essentially the same results except for variations in the fourth decimal place of the value of γ^*. This is due to the improved accuracy of the solution as a result of the numerical methods used here. The primary emphasis of this investigation is centered on determining the onset of flow instabilities at a theoretical critical Reynolds number near 1150. However, solutions were obtained at other Reynolds numbers to show stability trends and are presented in tabular and plotted form.

Stability data for angular wave number n = 0 for various combinations of Re and α are given in Table 6-1. The least stable eigenvalue solution of the governing equation is based on a computational mesh of N = 50. Table entries which contain dashes indicate that the solution of the governing equations was not sufficiently accurate to be included as valid data. Two facts become very clear in examining Table 6-1. First of all, the flow is stable at all Re and α for n = 0. Secondly, there is an asymptotic trend towards neutral stability for increasing Reynolds numbers. This second result agrees with experimental observations in that increasing Reynolds number has a destabilizing effect on the flow. It can also be seen that increasing the axial wave number has

Table 6-1. Stability Data for Angular Wave Number n = 0

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	-0.0879	-0.1544	-0.2360	-0.3399	-0.5152	-0.8665	-1.7770	-4.7194
575	-0.0459	-0.1196	-0.1685	-0.2429	-0.3614	-0.5831	-1.1122	-2.7236
1150	-0.0229	-0.0836	-0.1188	-0.1703	-0.2498	-0.3891	-0.6937	-1.5548
2300	-0.0115	-0.0591	-0.0838	-0.1197	-0.1737	-0.2633	-0.4429	-0.9099
4600	-0.0057	-0.0418	-0.0592	-0.0842	-0.1212	-0.1796	-0.2871	-
9200	-0.0029	-0.0295	-0.0418	-0.0593	-0.0846	-0.1225	-	-
18400	-0.0014	-0.0209	-0.0295	-0.0416	-0.0586	-	-	-

Note: Values are based on 50 mesh points.

a stabilizing effect on the flow. Normalized axial perturbation velocity plots versus pipe radius were made at various Re and α to examine the behavior of the perturbation quantities corresponding to the least stable eigenvalue solution. Figures 6-1 through 6-9, on the following pages, are representative of the behavior of the axial perturbation velocity. The activity is concentrated near the pipe axis for n = 0 and dies out quickly as r approaches 1.0. Additionally, the perturbation velocity plots were made in order to determine if mesh fineness was sufficient to obtain an accurate solution of the equations. Trial calculations were made for uniform mesh sizes up to 100 where the numerical results did not vary appreciably from those reported here.

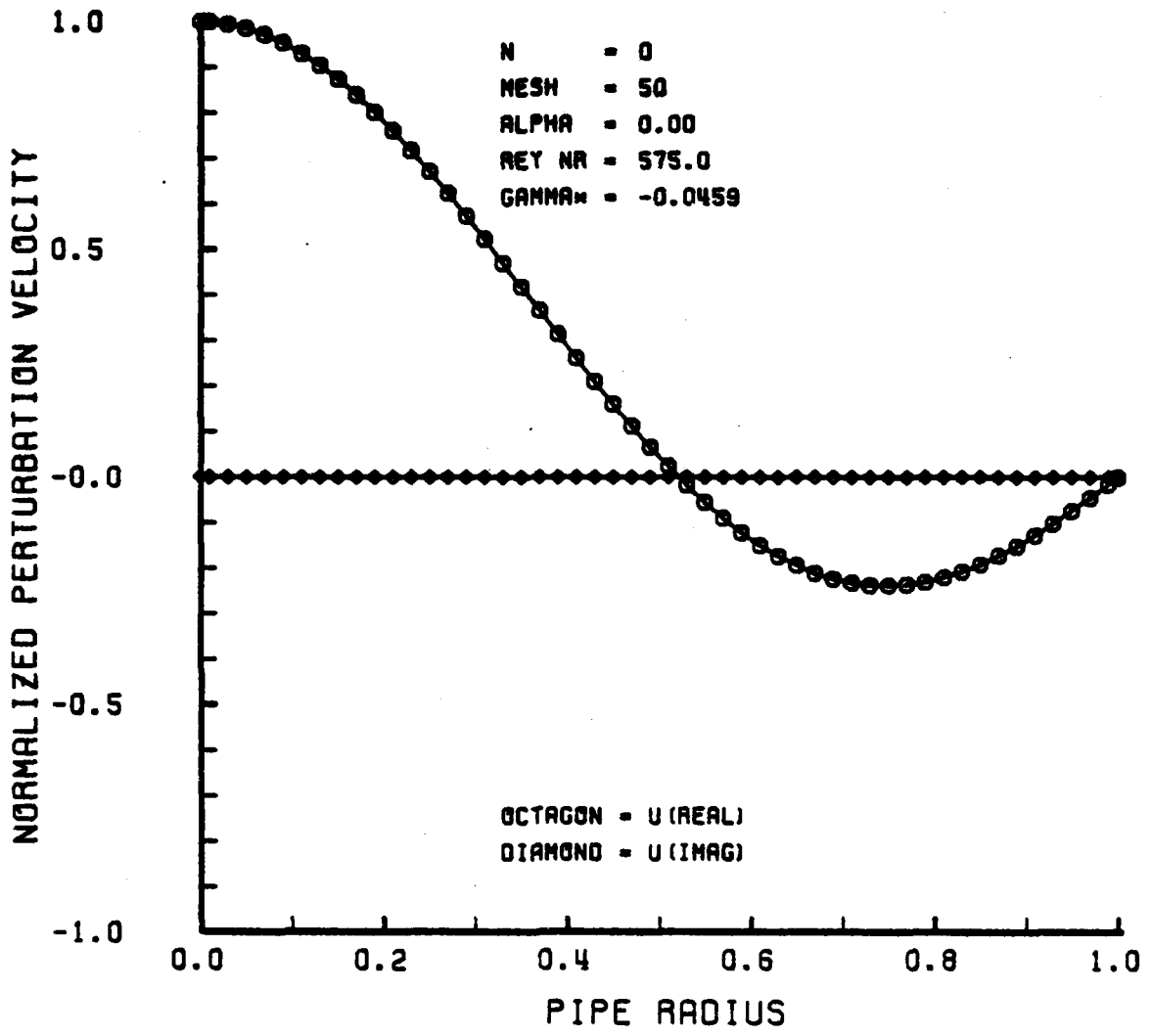
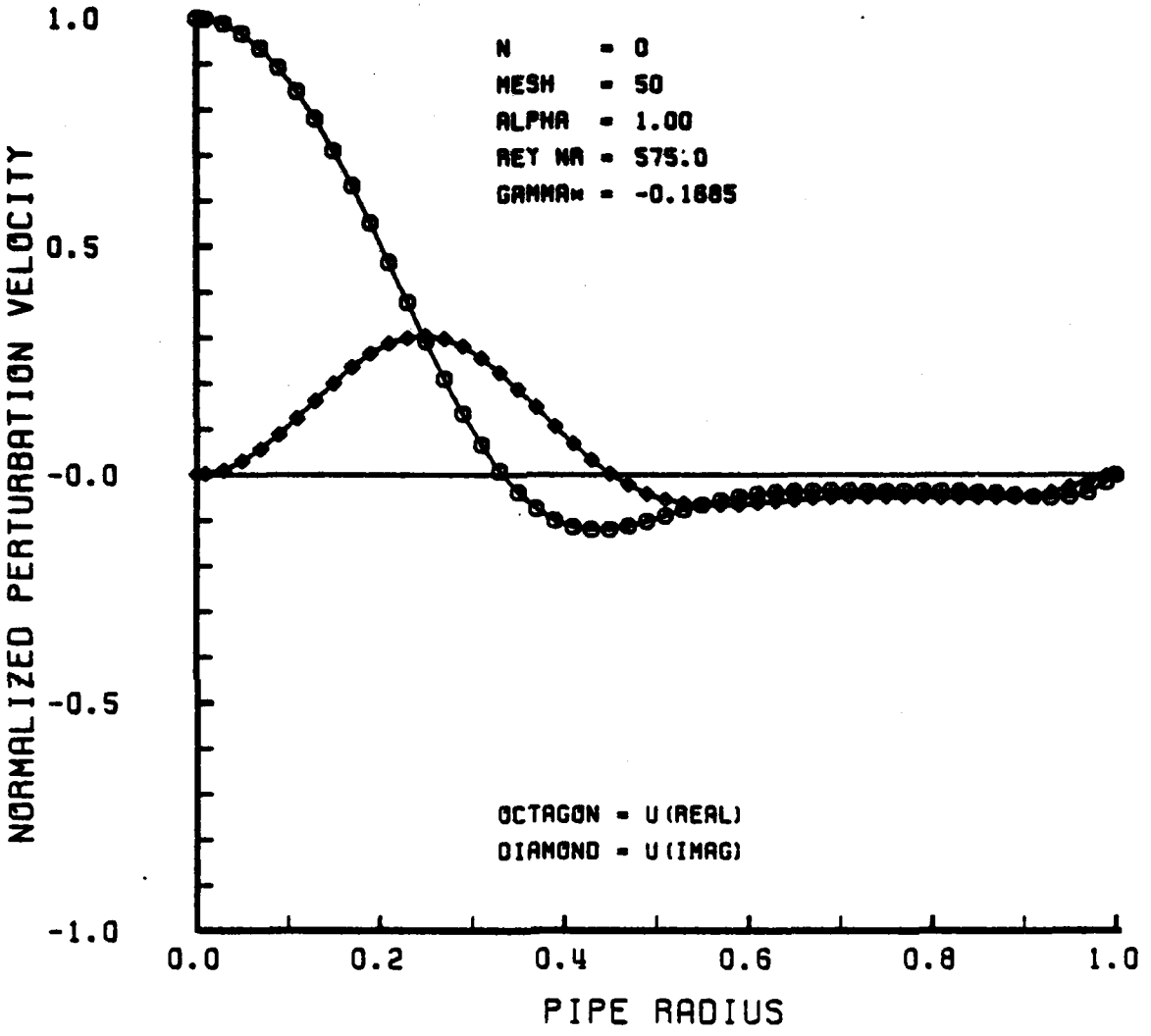

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 0.00
REY NR = 575.0
GAMMA* = -0.0459
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-1


NORMALIZED PERTURBATION VELOCITY VS RADIUS
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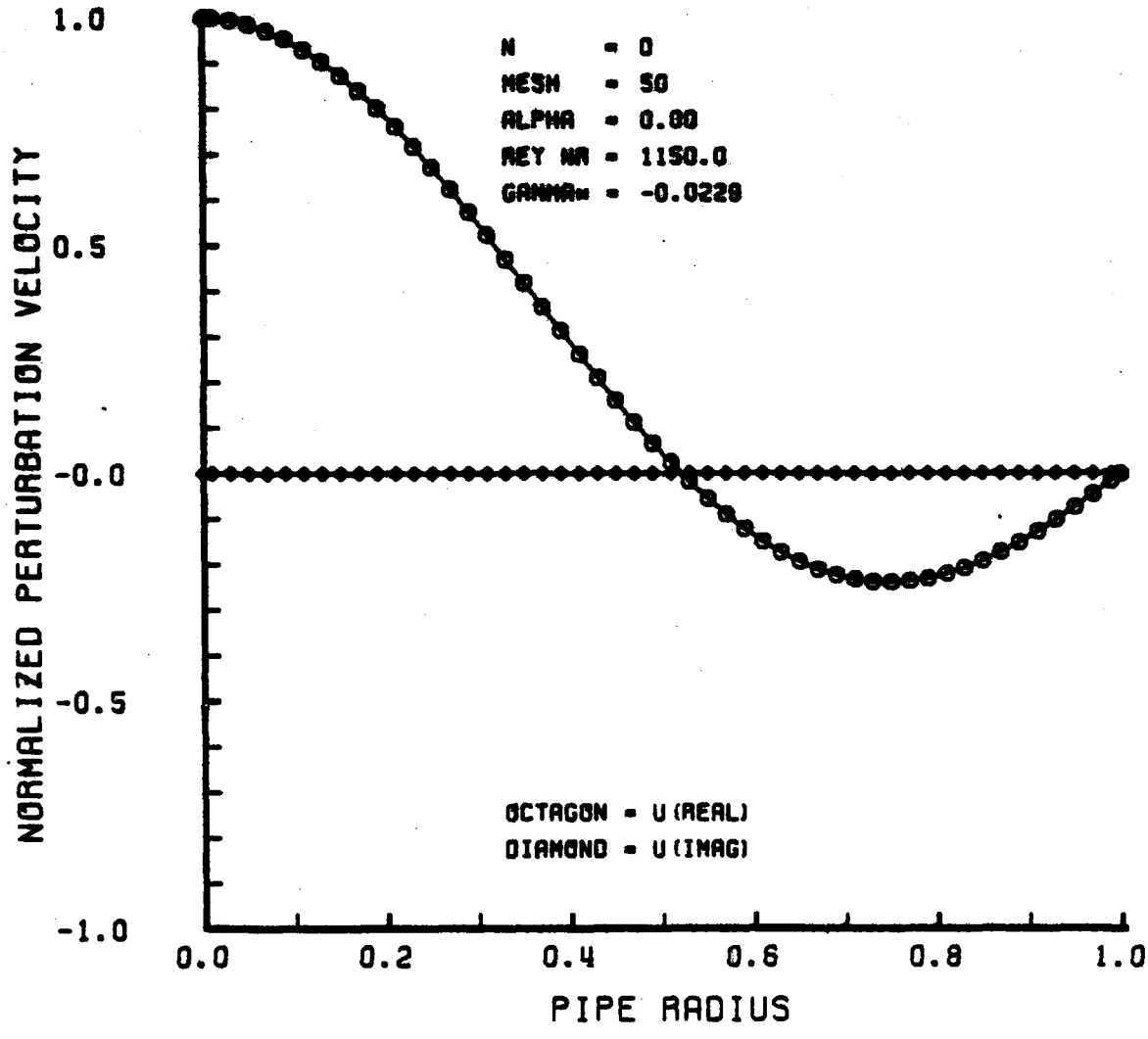
MESH = 50
ALPHA = 1.00
REY NR = 575.0
GAMMAw = -0.1685
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-2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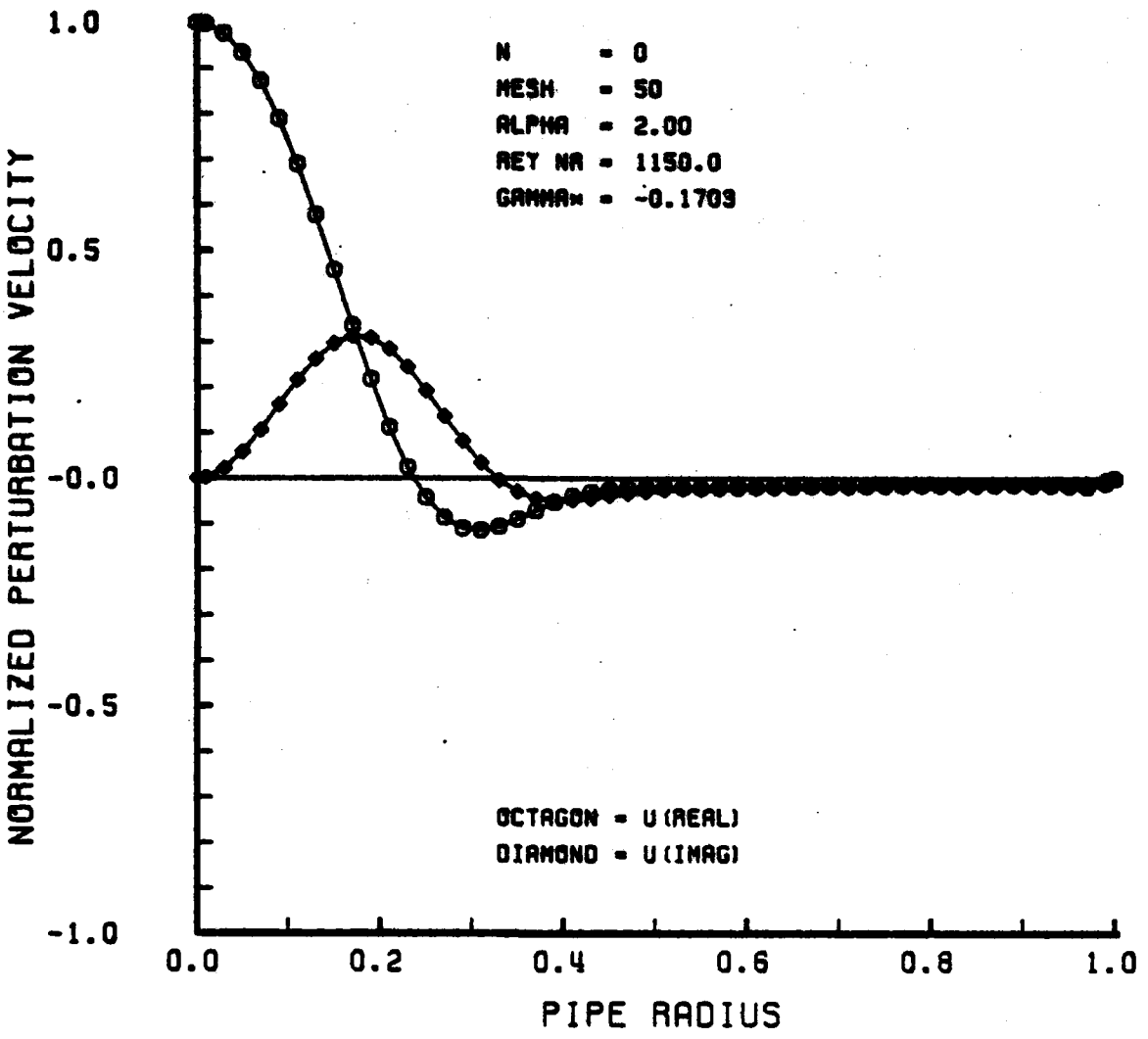
MESH = 50
ALPHA = 0.00
REY NR = 1150.0
GAMMA = -0.0228
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-3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N = 0
MESH = 50
ALPHA = 2.00
REY NR = 1150.0
GAMMA* = -0.1703
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-4

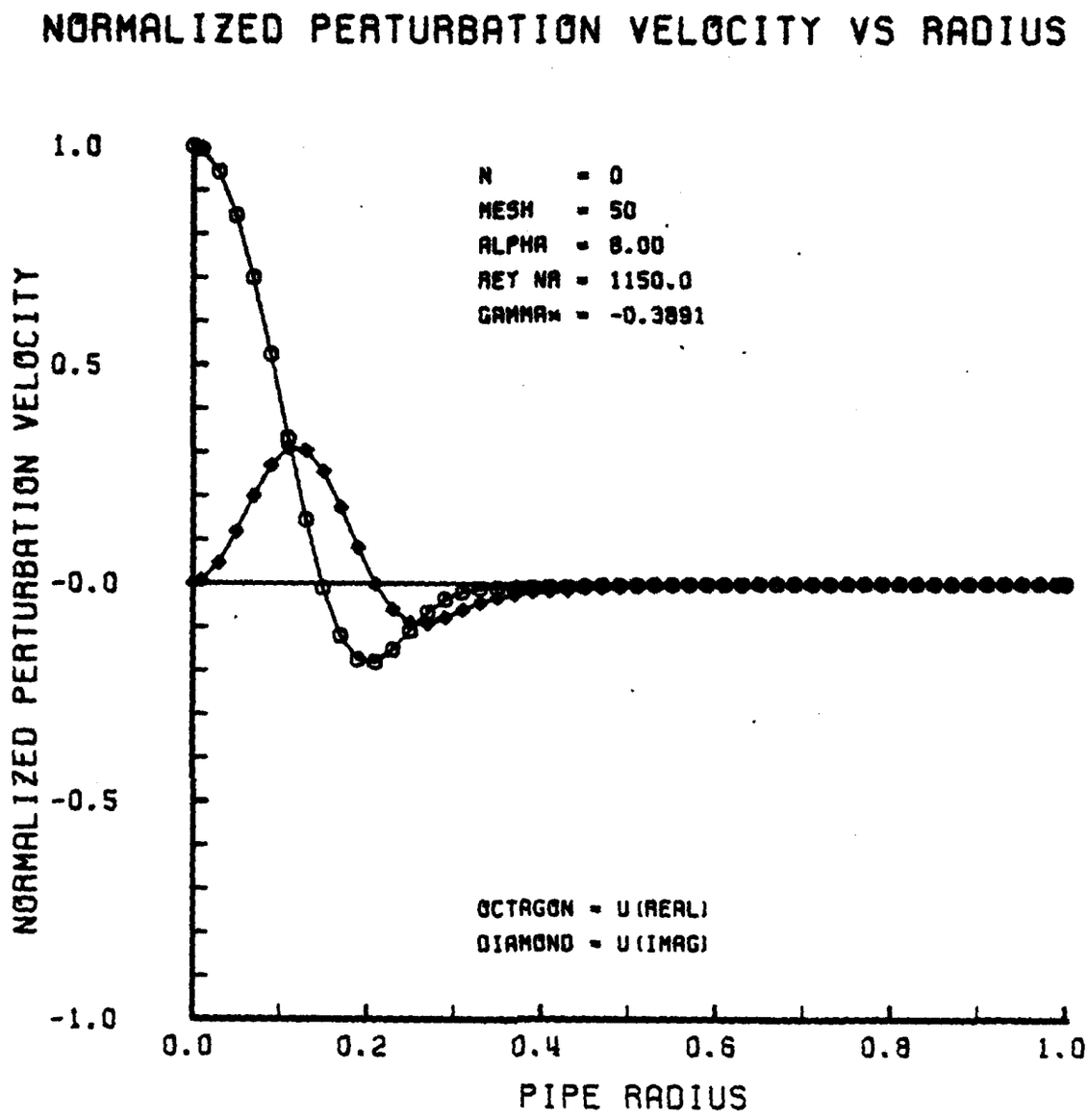

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 8.00
REY NR = 1150.0
GAMMA* = -0.3891
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-5

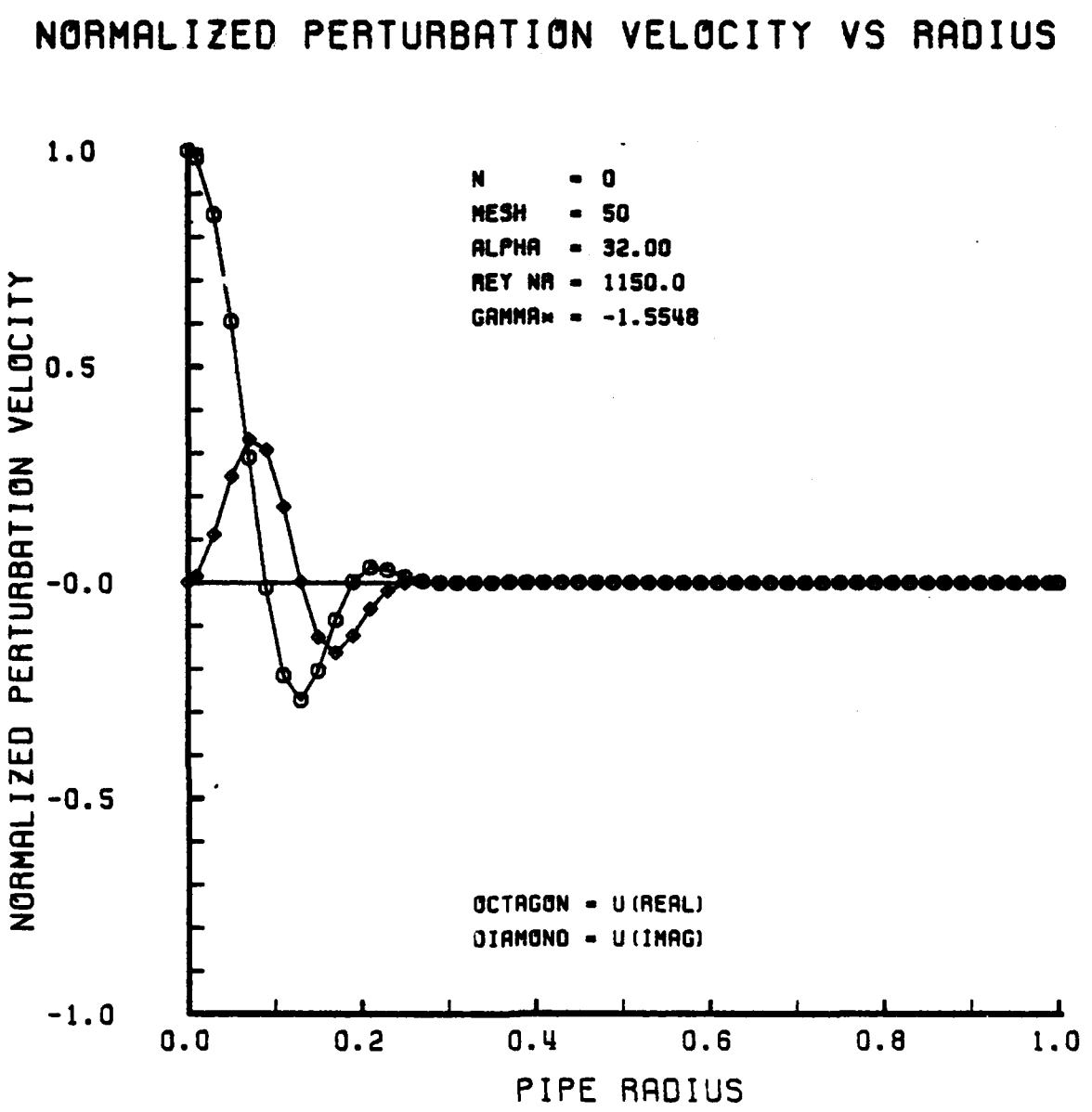
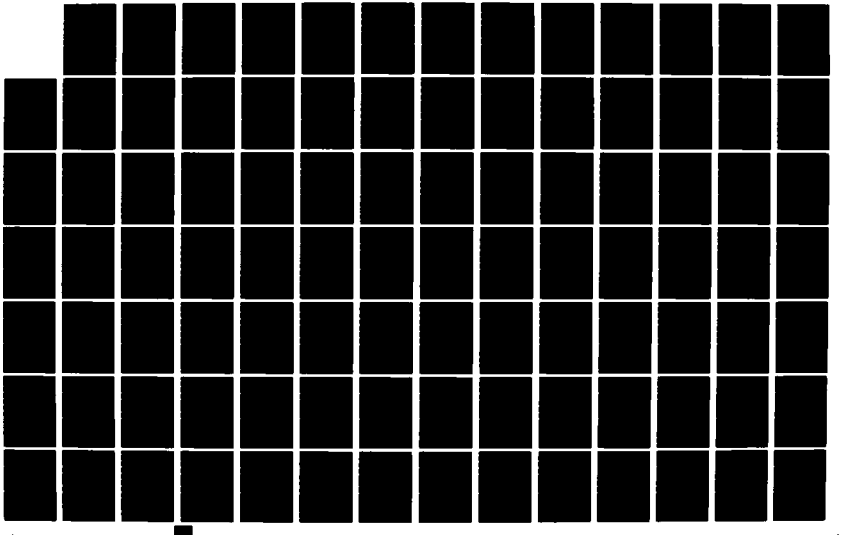
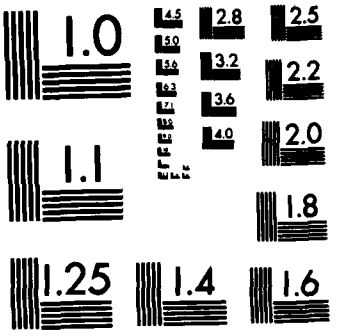

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 32.00
REY NR = 1150.0
GAMMA* = -1.5548
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-6

AD-A126 767 A NUMERICAL ANALYSIS OF PIPE FLOW STABILITY(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA D B WALLACE DEC 82 2/3

UNCLASSIFIED F/G 20/4 NL





MICROCOPY RESOLUTION TEST CHART
NATIONAL BUREAU OF STANDARDS-1963-A

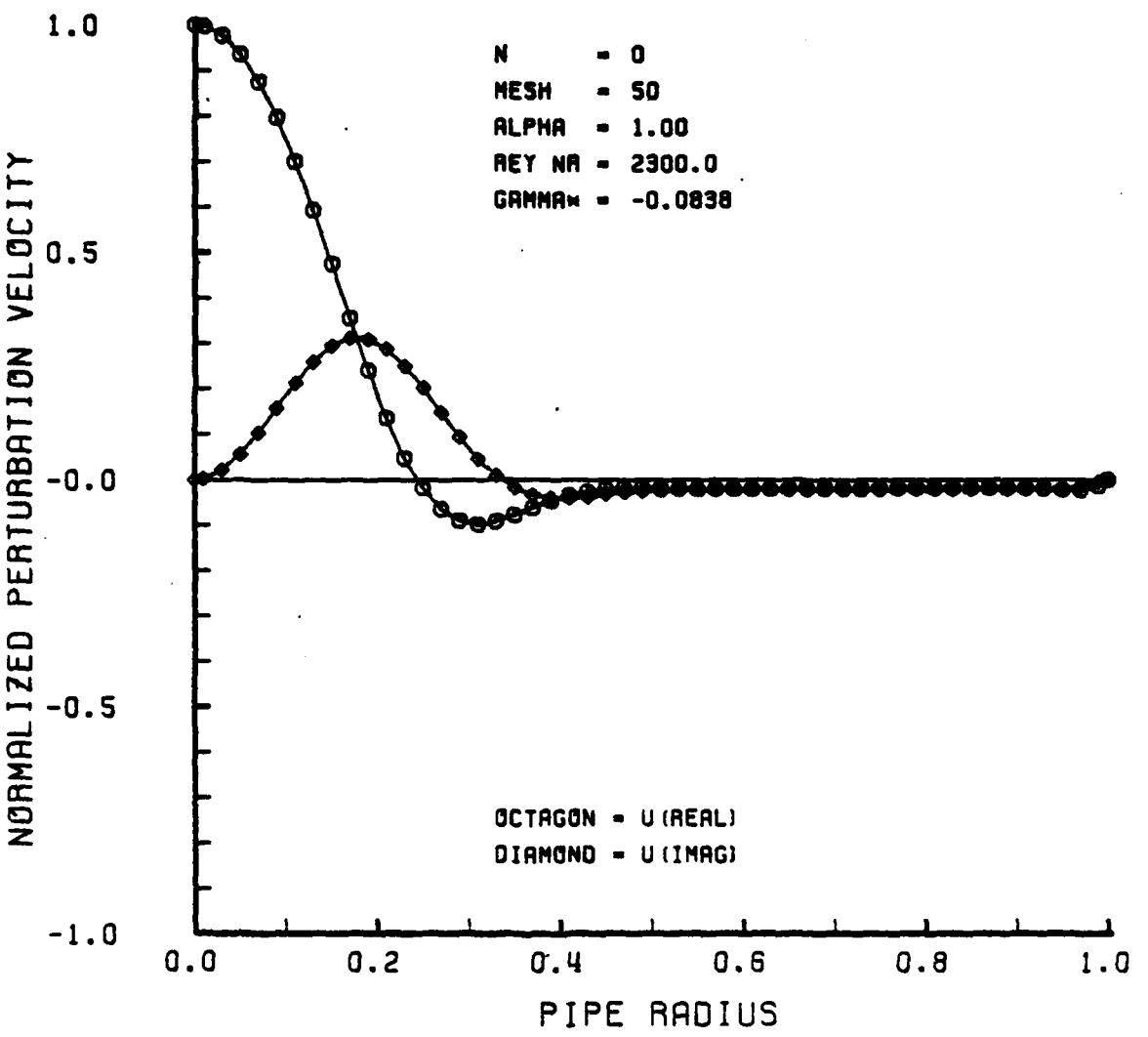

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 2300.0
GAMMA* = -0.0838
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-7

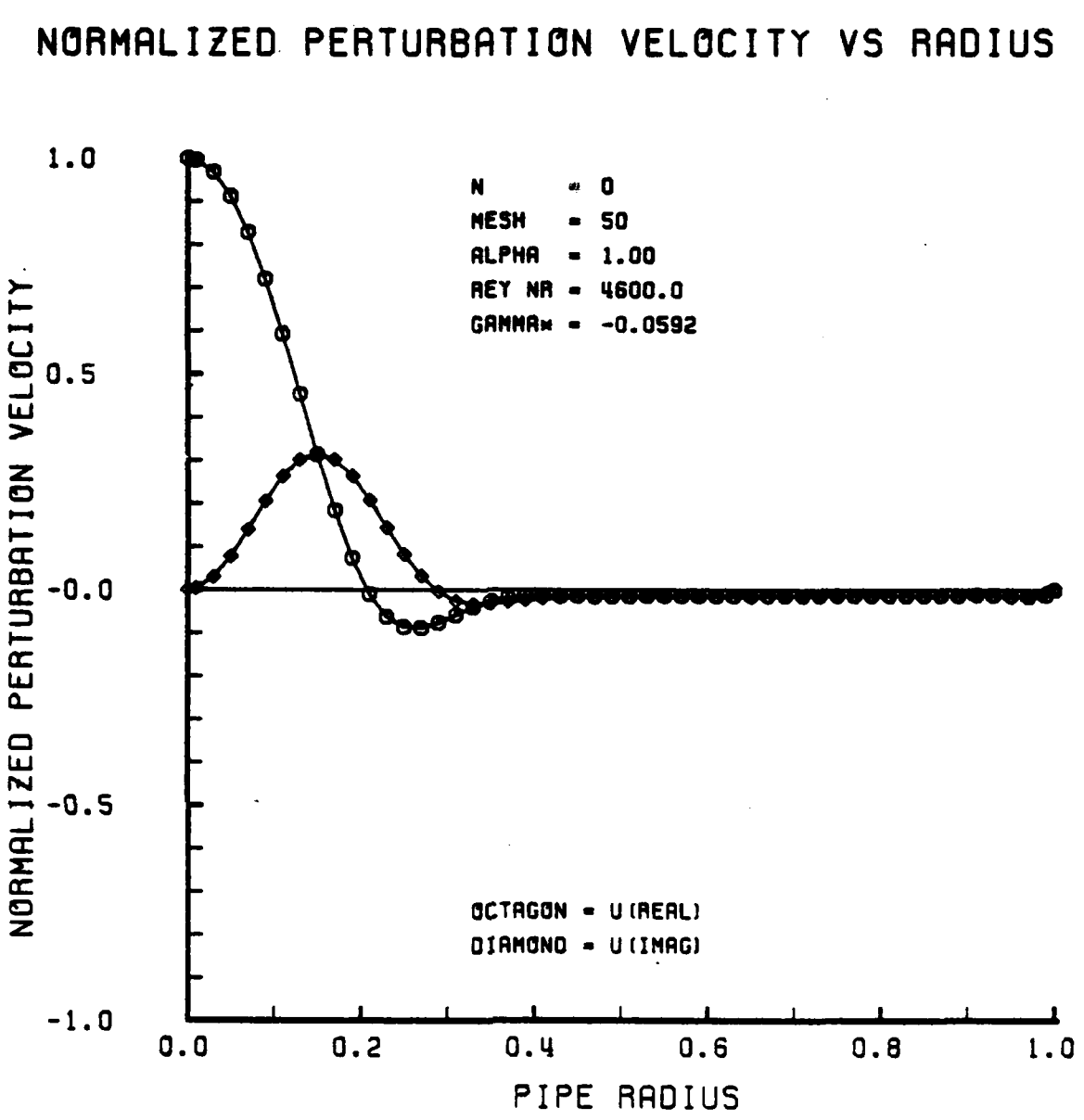

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 4600.0
GAMMA* = -0.0592
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-8

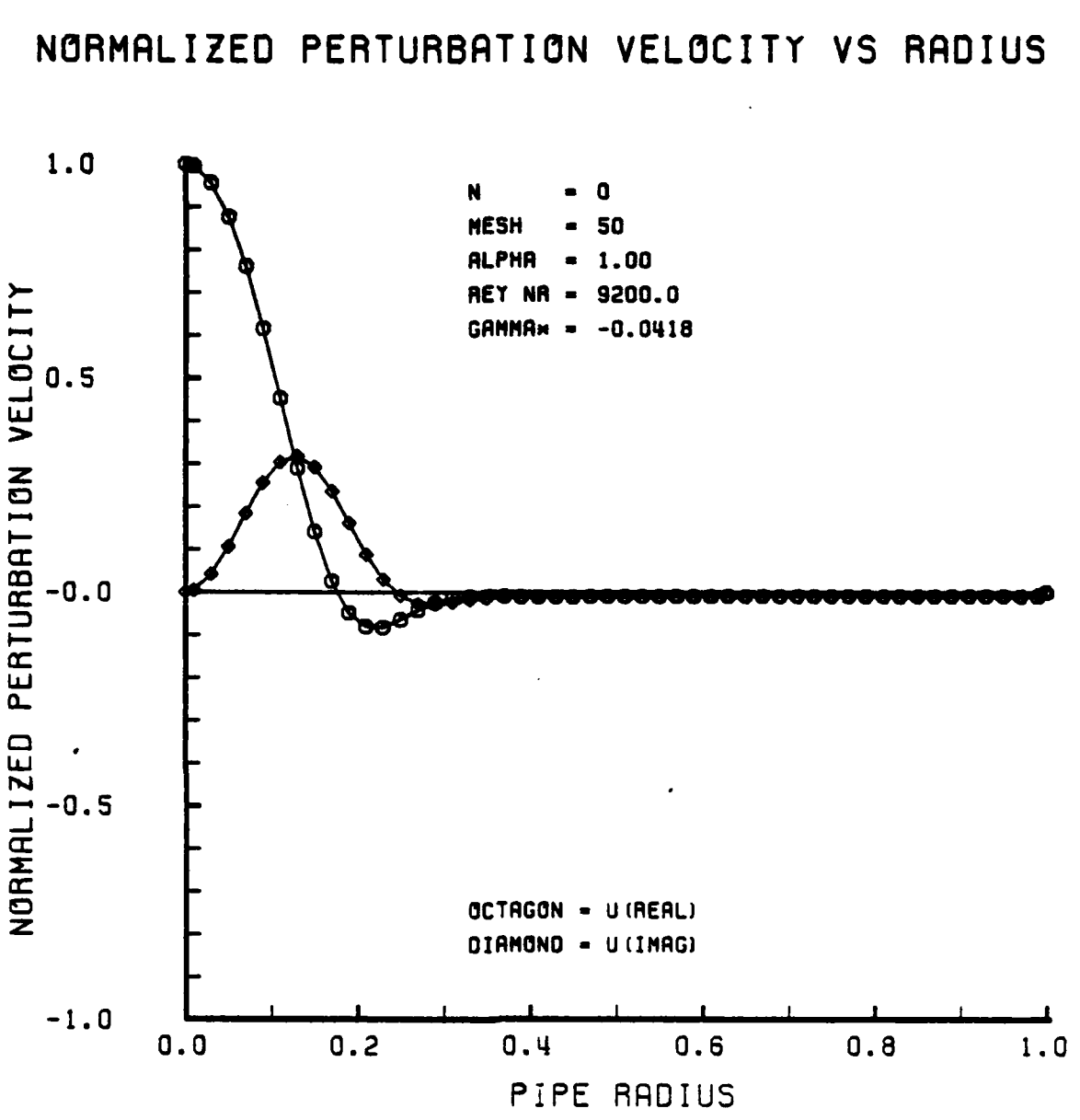

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 1.00
REY NR = 9200.0
GAMMA* = -0.0418
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-9

B. RESULTS FOR n = 1

The numerical results for angular wave number n = 1 indicate that flow instabilities indeed exist. There are four factors that may possibly account for the flow instabilities; (1) the complicated nature of the coupled governing equations for $n = 1$, (2) the rigorous way in which the boundary conditions at the axis were derived and enforced, (3) the introduction of additional parameters into the eigensystem, and (4) the appearance of the eigenvalue in a pair of the boundary condition equations. Table 6-2 summarizes the stability data for n = 1 over a wide range of Reynolds numbers and axial wave numbers. Once again the dashed table entries indicate inaccurate results. Notice that a mesh size of n = 48 was used for this case. This was necessary because of programming constraints that required the eigensystem to be less than or equal to 100 X 100. It can be seen that flow instabilities exist at all Reynolds numbers. This occurs coincidentally with small values of α. More detailed stability data is presented in Table 6-3 for axial wave numbers $0 \leq \alpha \leq 1$. It is clear here that the flow becomes more unstable for very small values of $\alpha < 1$. The stability data presented in Tables 6-2 and 6-3 is more easily interpreted as Reynolds number contours in the plots of GAMMA* vs. alpha. Figure 6-10 shows Reynolds number contours over a large range of alpha. It can be clearly seen that the flow is unstable at all Reynolds numbers for small values of axial wave numbers. Figure 6-11 shows a more detailed plot of the Reynolds number contours over a small range of α. Here it can be seen that the flow becomes more unstable for decreasing values of Reynolds numbers! It can

Table 6-2. Stability Data for Angular Wave Number n = 1

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	0.0606	0.0626	-0.0215	-0.2883	-0.3734	-0.7623	-1.3222	-
575	0.0316	0.0270	-0.0601	-0.1983	-0.2530	-0.4638	-1.1246	-0.7546
1150	0.0158	0.0026	-0.0876	-0.1362	-0.1723	-0.2805	-0.7067	-
2300	0.0079	-0.0131	-0.0675	-0.0947	-0.1205	-0.1745	-0.4004	-
4600	0.0039	-0.0237	-0.0476	-0.0664	-0.0860	-0.1117	-0.2327	-
9200	0.0020	-0.0239	-0.0336	-0.0467	-0.0617	-0.0736	-	-
18400	0.0010	-0.0169	-0.0238	-0.0328	-0.0436	-0.0497	-	-

Note: Values are based on 48 mesh points.

Table 6-3. Stability Data for Angular Wave Number n = 1

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=0.10	α=0.15	α=0.20	α=0.50	α=0.75	α=1.0
300	0.0606	0.0657	0.0718	0.0743	0.0743	0.0626	0.0322	-0.0215
575	0.0316	0.0374	0.0394	0.0395	0.0398	0.0270	-0.0057	-0.0601
1150	0.0158	0.0198	0.0204	0.0208	0.0202	0.0026	-0.0324	-0.0876
2300	0.0079	0.0103	0.0105	0.0096	0.0081	-0.0131	-0.0507	-0.0675
4600	0.0039	0.0053	0.0044	0.0028	0.0006	-0.0237	-0.0413	-0.0476
9200	0.0020	0.0022	0.0006	-0.0016	-0.0042	-0.0239	-0.0292	-0.0336
18400	0.0010	0.0003	-0.0018	-0.0044	-0.0074	-0.0169	-0.0206	-0.0238

Note: Values are based on 48 mesh points.

also be seen here that the flow becomes even more unstable for small values of α only slightly larger than zero. The reason for these results cannot be readily explained but it is obvious that they are in direct conflict with observed experimental results. When $\alpha > 1$, however, the contours show an asymptotic trend toward neutral stability for increasing Reynolds numbers; in other words, increasing Reynolds number has a destabilizing effect on the flow which agrees with experimental results. A slightly different method of presenting the data can be seen in Figure 6-12, as alpha contours in the plot of GAMMA* vs. Reynolds number. This plot further emphasizes the results discussed above; the flow is unstable at all Reynolds numbers for small alpha, increasing Reynolds number has a destabilizing effect on the flow and increasing axial wave number has a stabilizing effect.

Normalized perturbation velocity plots were also produced for this case and show some interesting trends. The behavior of the axial perturbation velocity for selected values of Re and α appear in Figures 6-13 through 6-23 on the following pages. Upon close examination of the perturbation velocity plots, it becomes apparent that the computational mesh is not sufficiently fine to adequately approximate the velocity function as α is increased. This problem will be discussed shortly.

It is interesting to note that the perturbation activity shifts from the pipe wall to the axis for increasing values of the axial wave number while Reynolds number remains constant. At small values of α, the activity originates at the wall. At larger values of α, the activity shifts to the axis, for the most part. At even larger values of α, the activity moves away from the axis to the region between the axis and

wall. However, some degree of activity remains at the wall for the last two cases. The change in the perturbation activity as described above can be seen in Figures 6-14 through 6-16 for Re = 575 and in Figures 6-19 through 6-21 for Re = 2300. The shift of perturbation activity to the axis with increasing axial wave number deserves comment here. Schlichting [Ref. 16] observed that the transition to turbulence is "characterized by the ..... appearance of self-sustaining turbulent flashes with emanate from fluid layers near the wall along the tube." Therefore, it appears that unstable flows are associated with perturbation activity near the wall. As the activity shifts away from the wall with increasing axial wave number, the flow becomes stable. Additionally, after the activity moves to the axis, some degree of activity remains at the wall. While the solution appears to remain stable, it is recommended that this peculiarity be examined further. The use of a finer mesh to resolve this activity at the wall would be useful here.

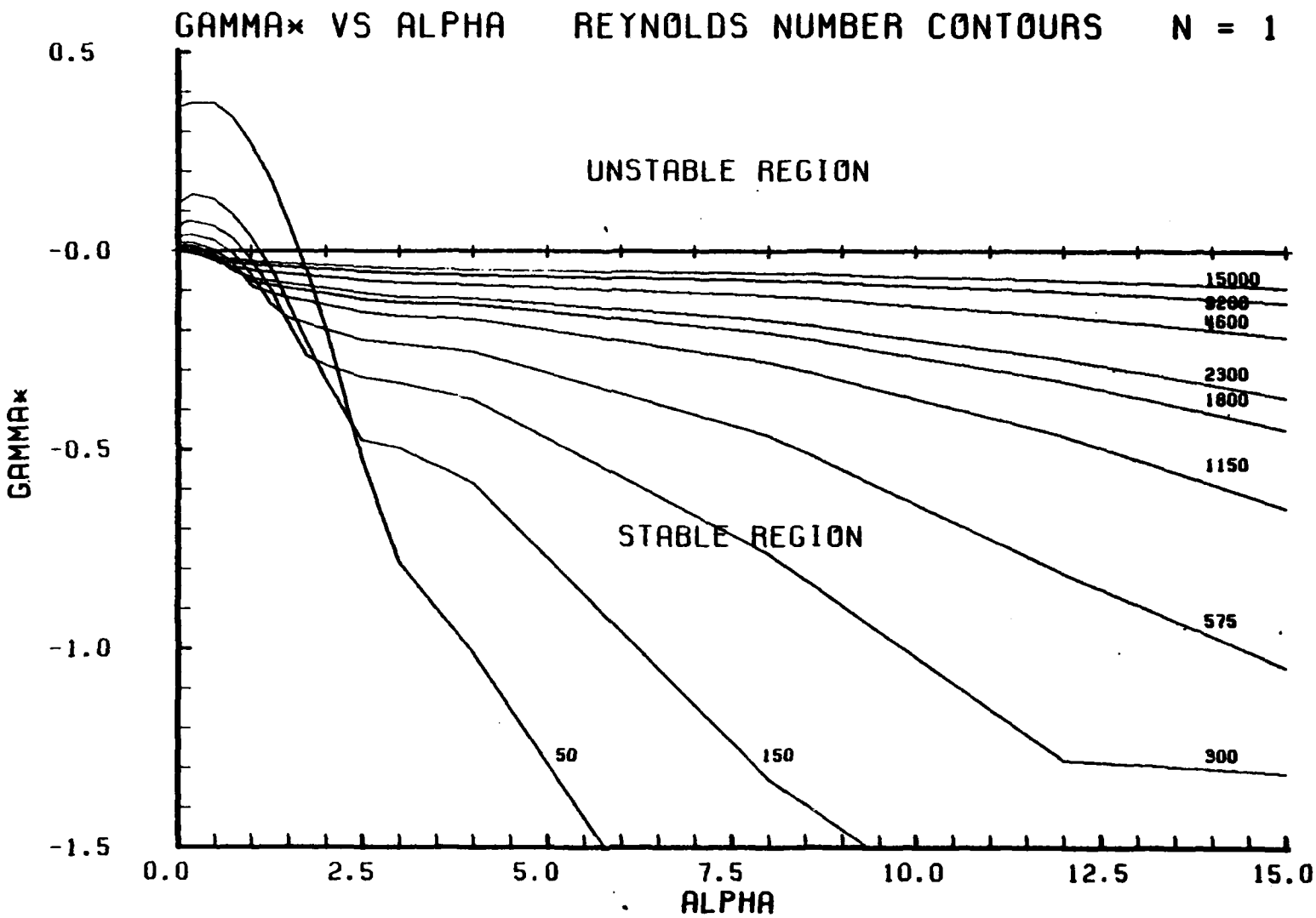

GAMMA× VS ALPHA REYNOLDS NUMBER CONTOURS N = 1
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
50
0.5
-0.0
-0.5
-1.0
-1.5
GAMMA×
0.0
2.5
5.0
7.5
10.0
12.5
15.0
ALPHA

Figure 6-10

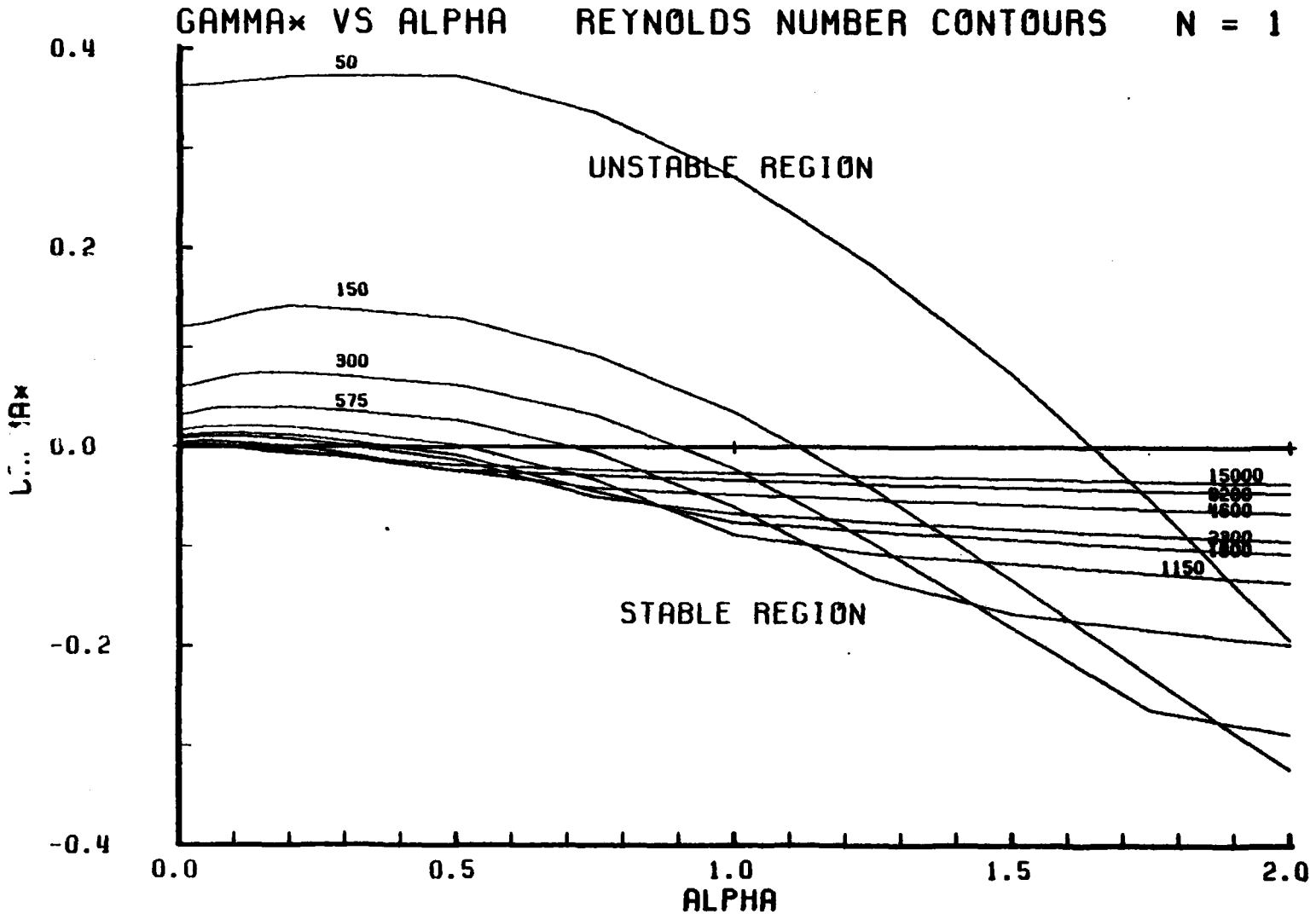

GAMMA× VS ALPHA REYNOLDS NUMBER CONTOURS N = 1
0.4
0.2
0.0
-0.2
-0.4
0.0
0.5
1.0
1.5
2.0
ALPHA
50
150
300
575
1150
15000
UNSTABLE REGION
STABLE REGION

Figure 6-11

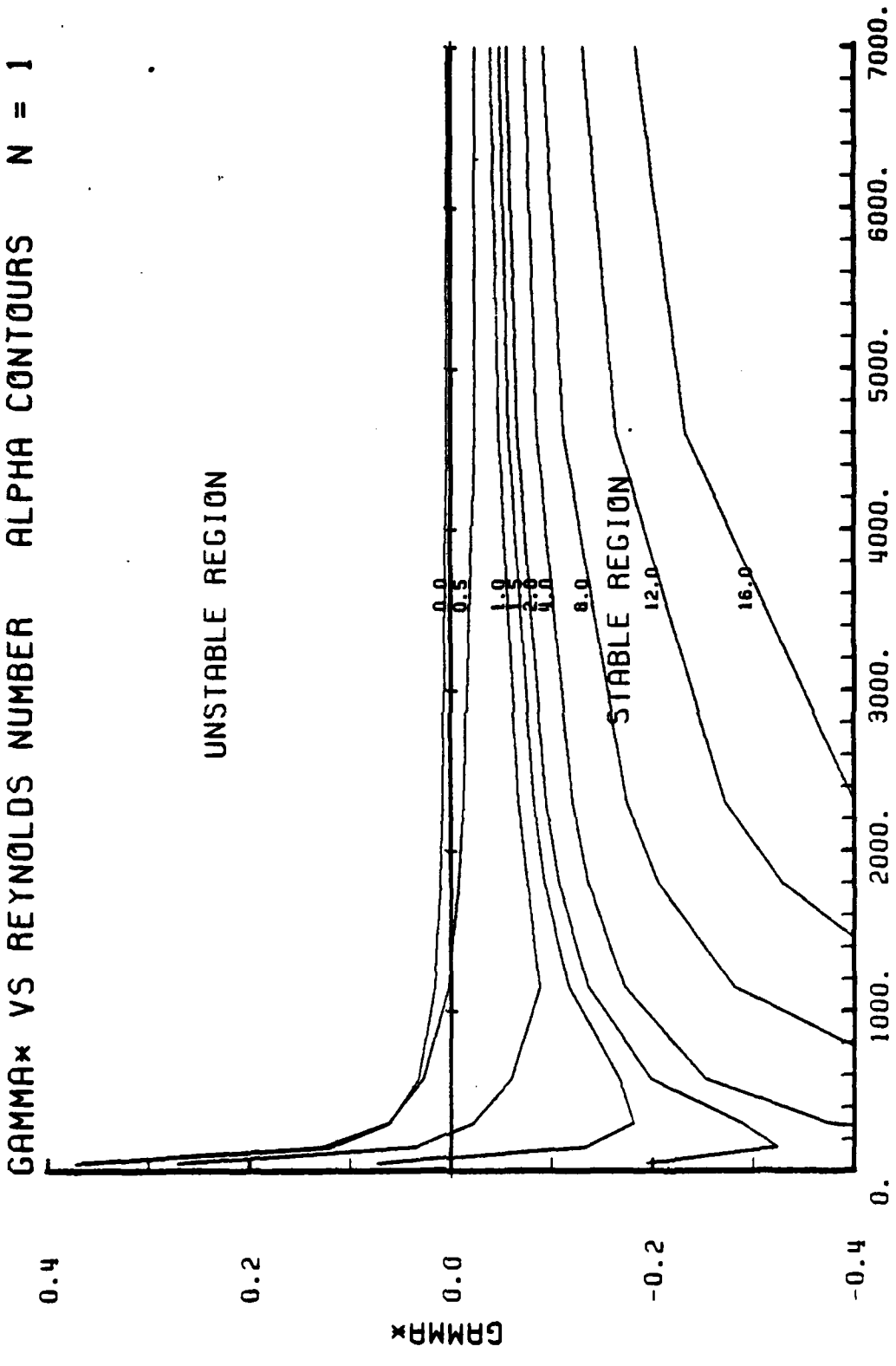

GAMMA× VS REYNOLDS NUMBER ALPHA CONTOURS N = 1
UNSTABLE REGION
STABLE REGION
0.0
0.5
1.0
1.5
2.0
4.0
8.0
12.0
16.0
0.4
0.2
0.0
-0.2
-0.4
GAMMA×
0.
1000.
2000.
3000.
4000.
5000.
6000.
7000.
REYNOLDS NUMBER

Figure 6-12

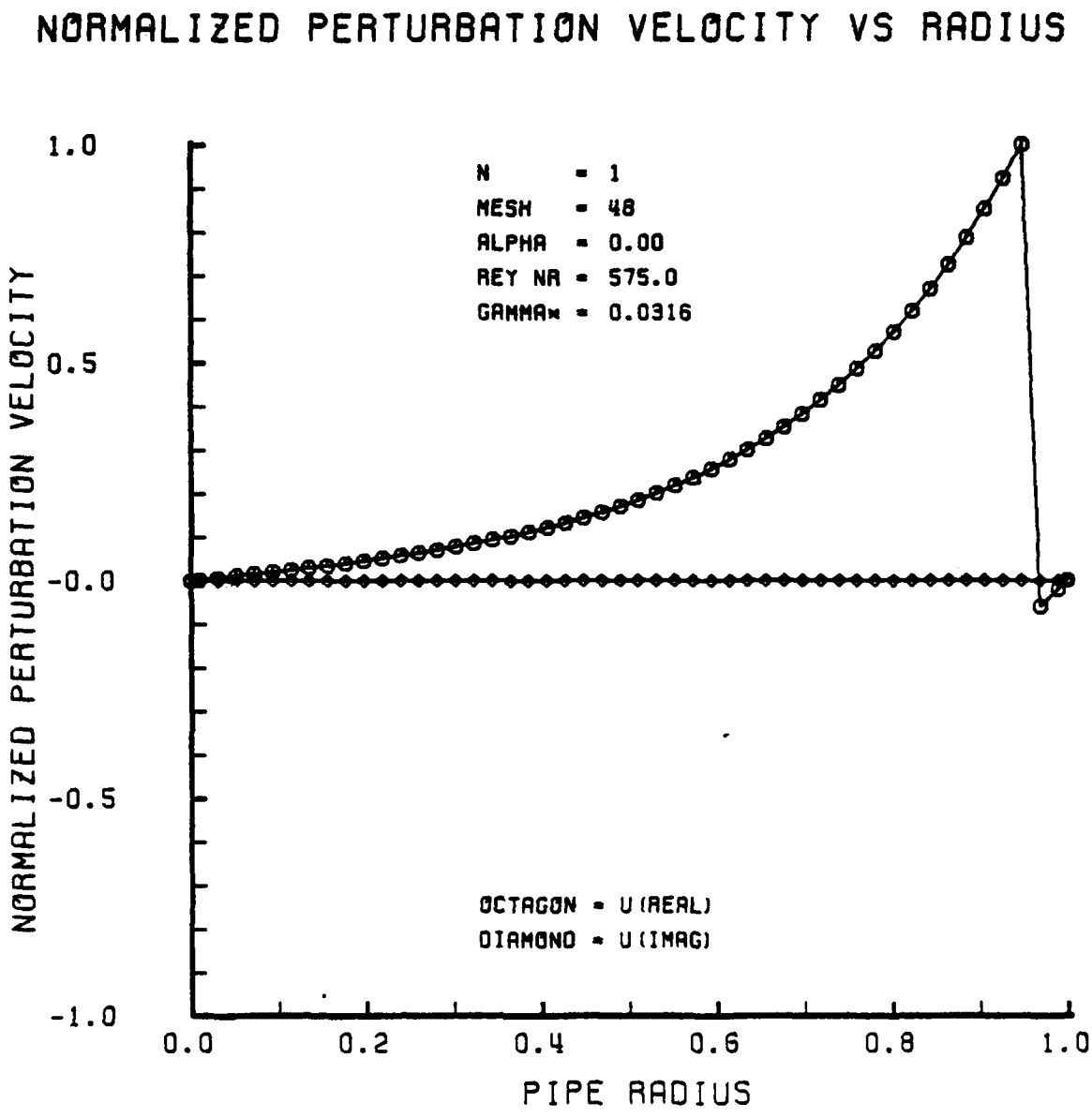

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.00
REY NR = 575.0
GAMMA* = 0.0316
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-13

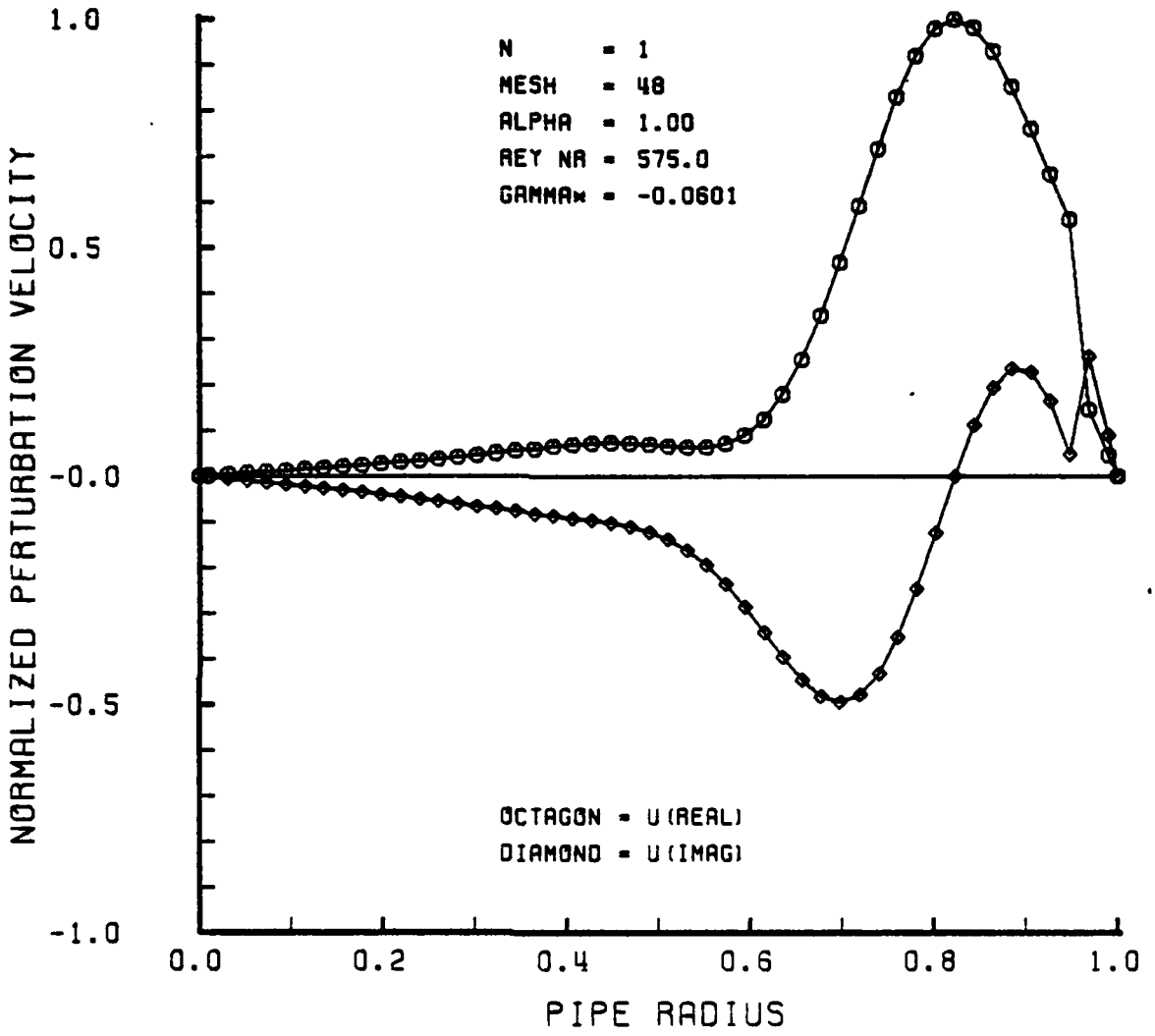

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 575.0
GAMMA* = -0.0601
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-14

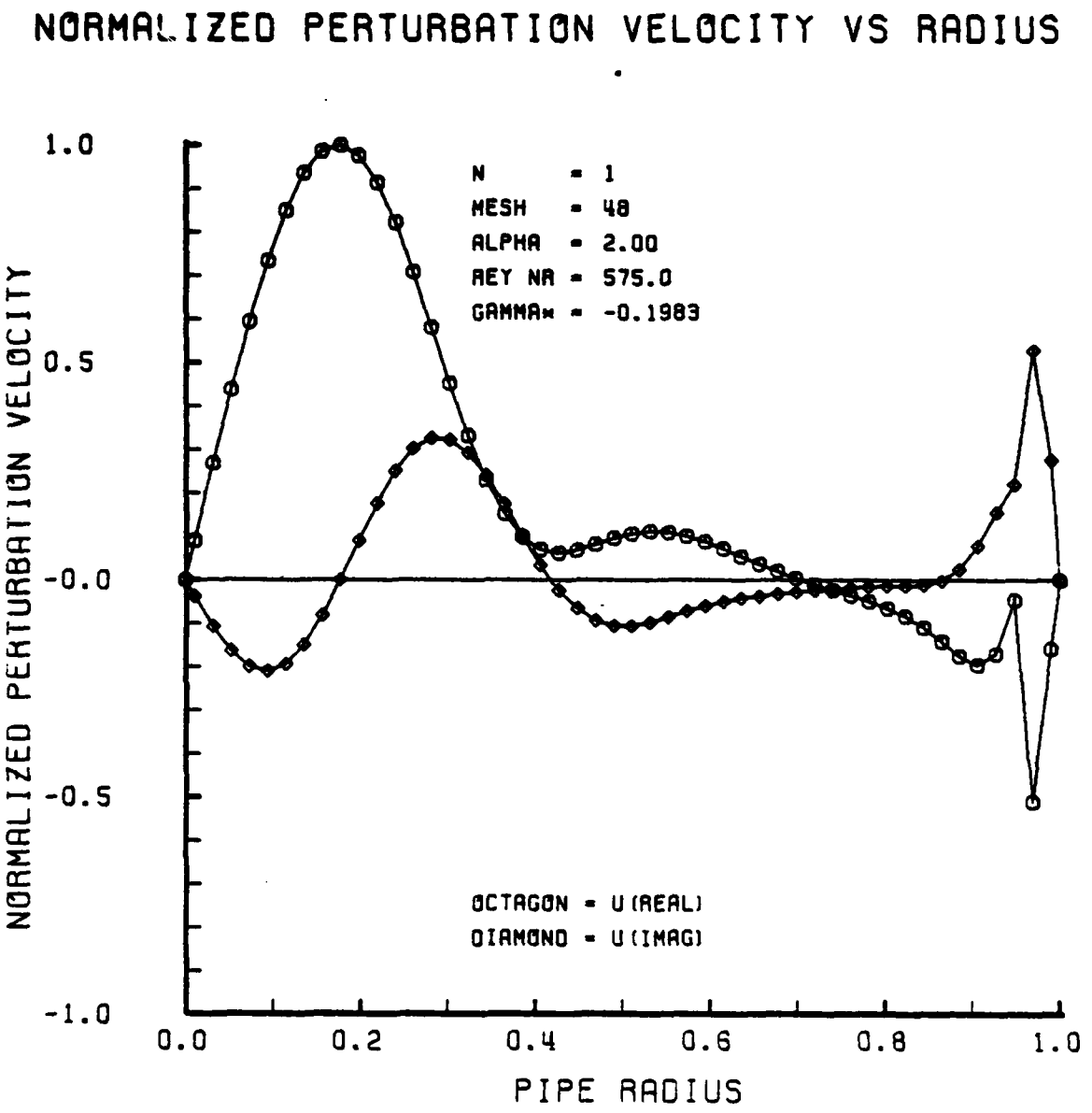

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 2.00
REY NR = 575.0
GAMMA= = -0.1983
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-15

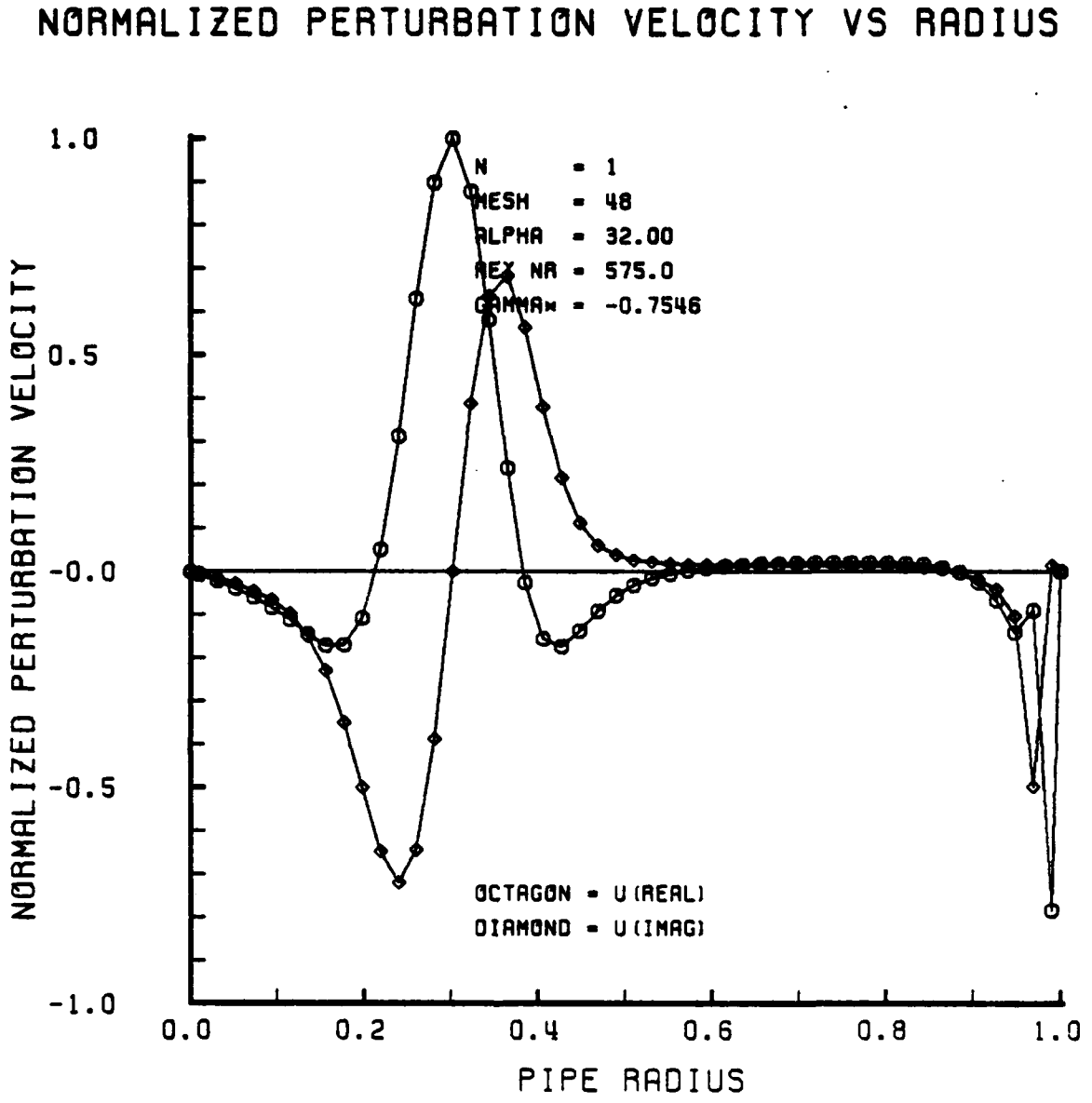

NORMALIZED PERTURBATION VELOCITY VS RADIUS
1.0
0.5
-0.0
-0.5
-1.0
NORMALIZED PERTURBATION VELOCITY
N = 1
MESH = 48
ALPHA = 32.00
REX NR = 575.0
GAMMAR = -0.7546
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-16

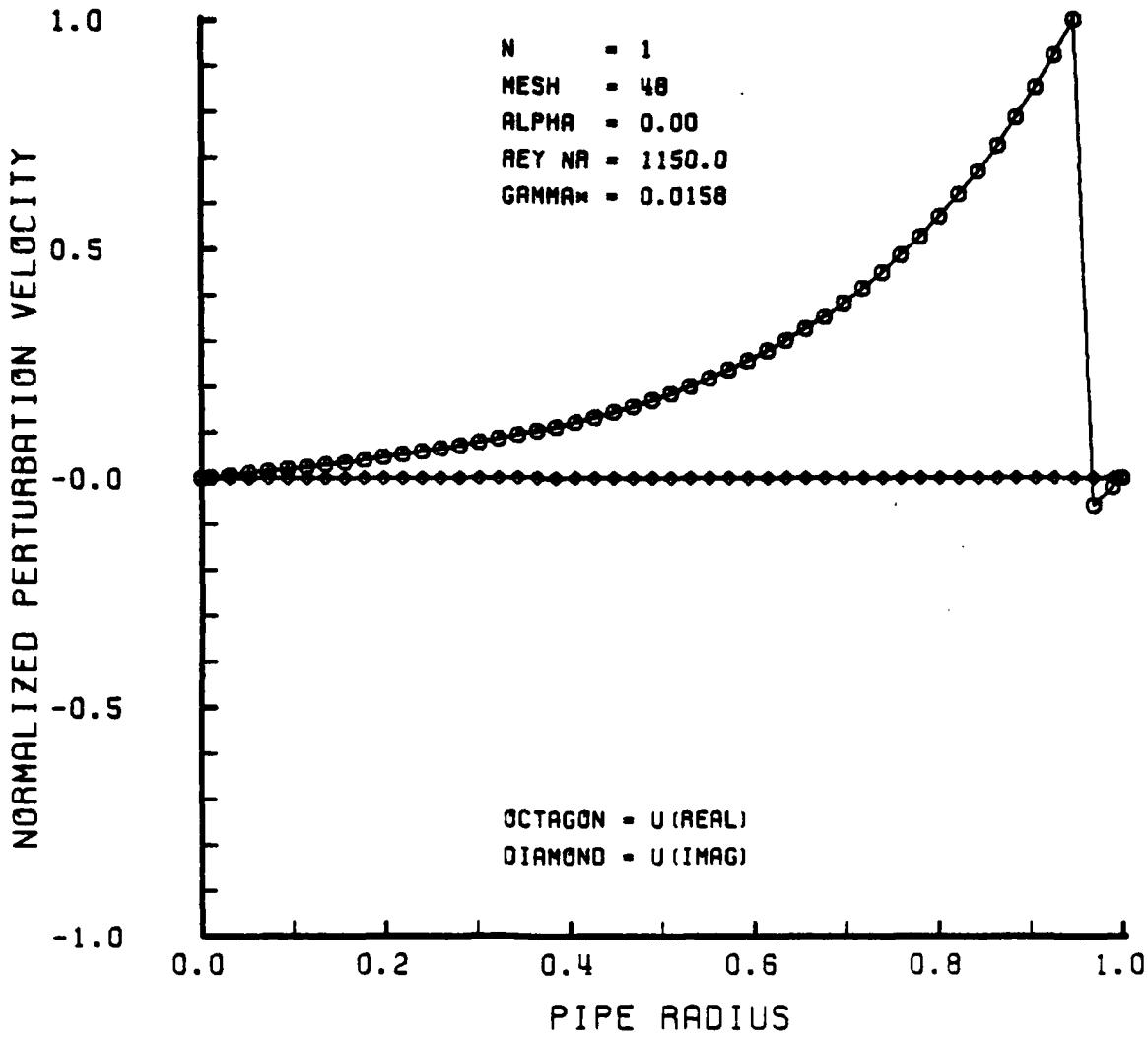

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.00
REY NR = 1150.0
GAMMA* = 0.0158
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-17

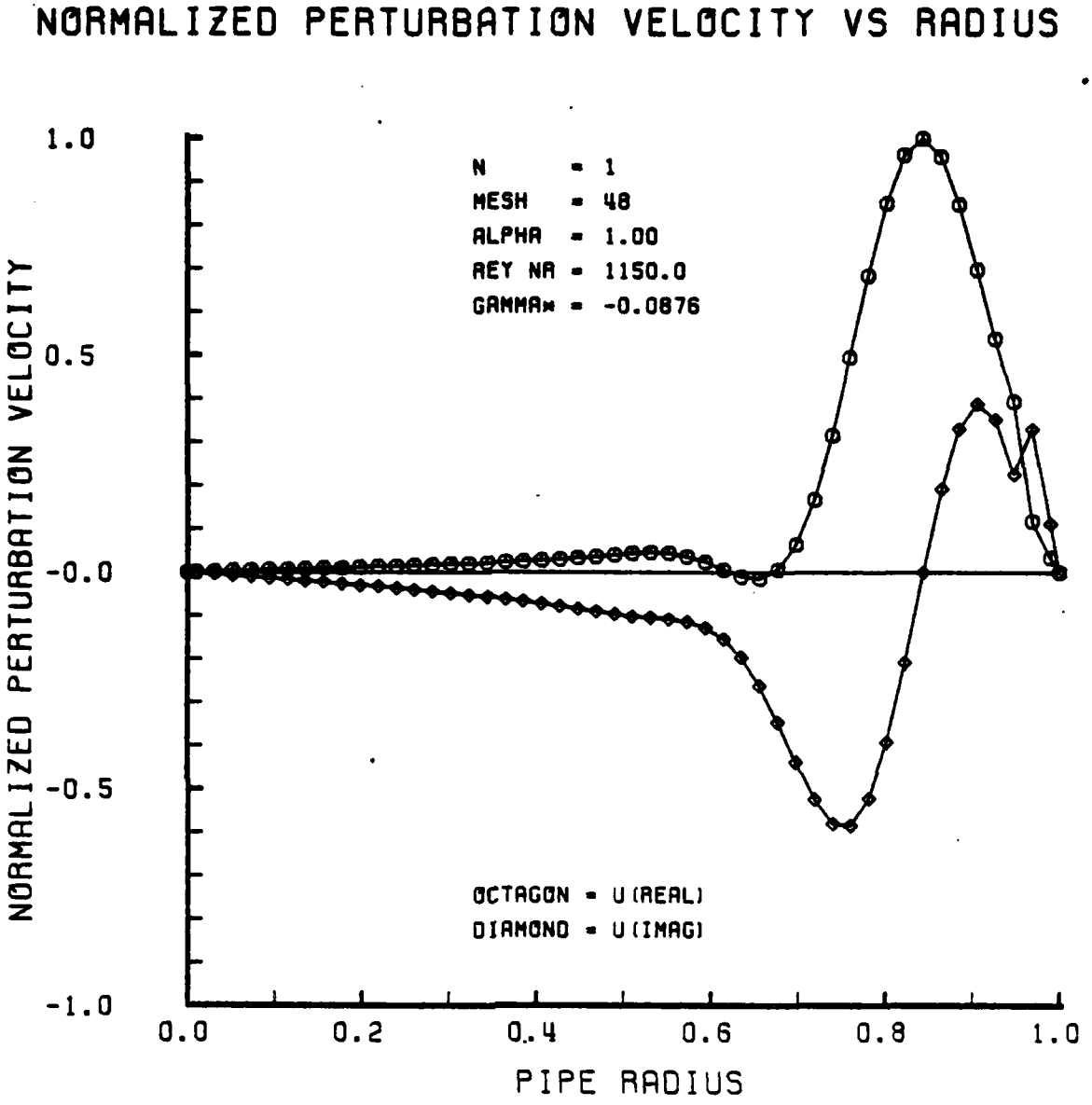

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 1150.0
GAMMA* = -0.0876
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-18

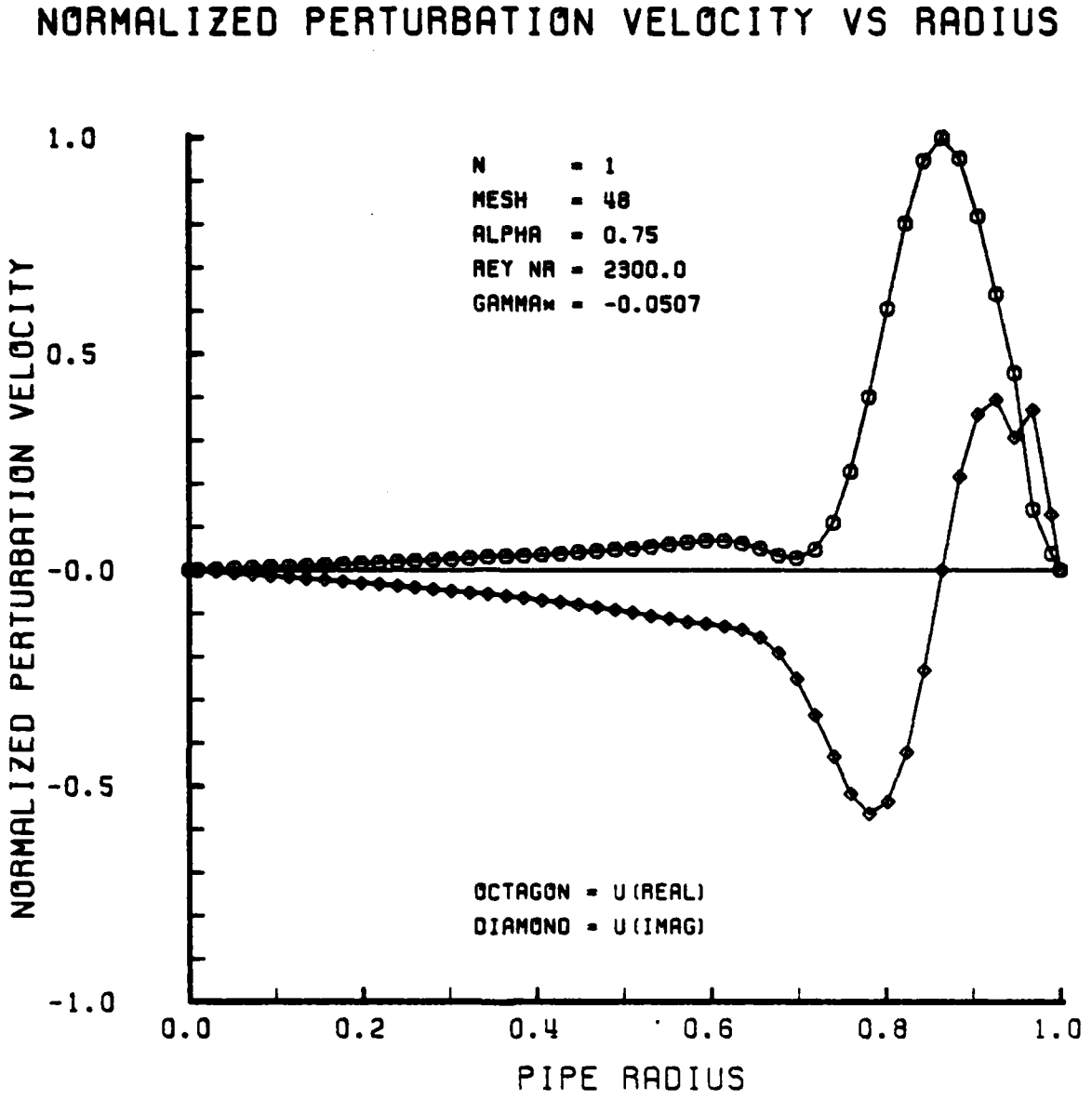

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 0.75
REY NR = 2300.0
GAMMA* = -0.0507
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-19

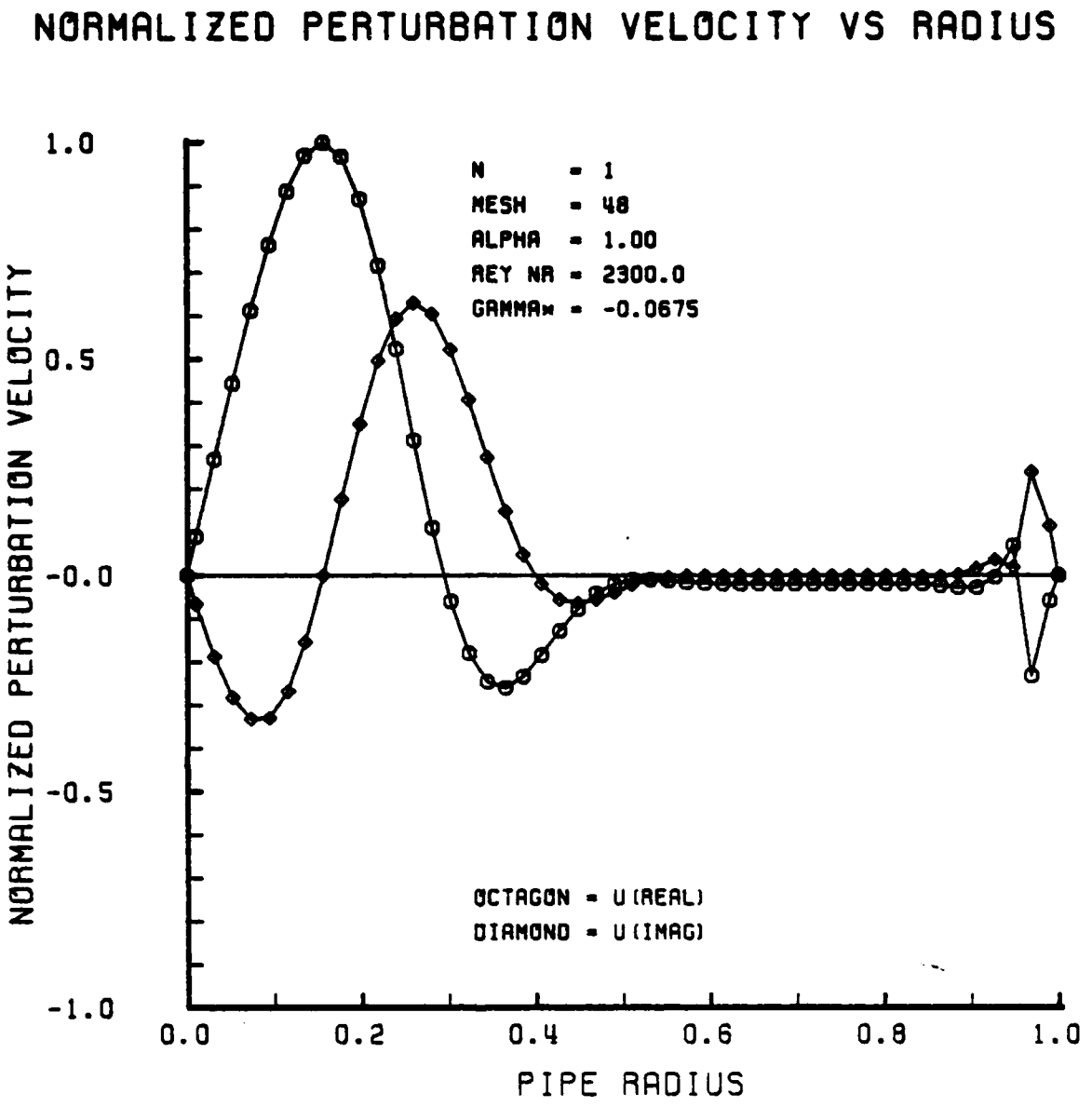

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 1.00
REY NR = 2300.0
GAMMA* = -0.0675
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-20

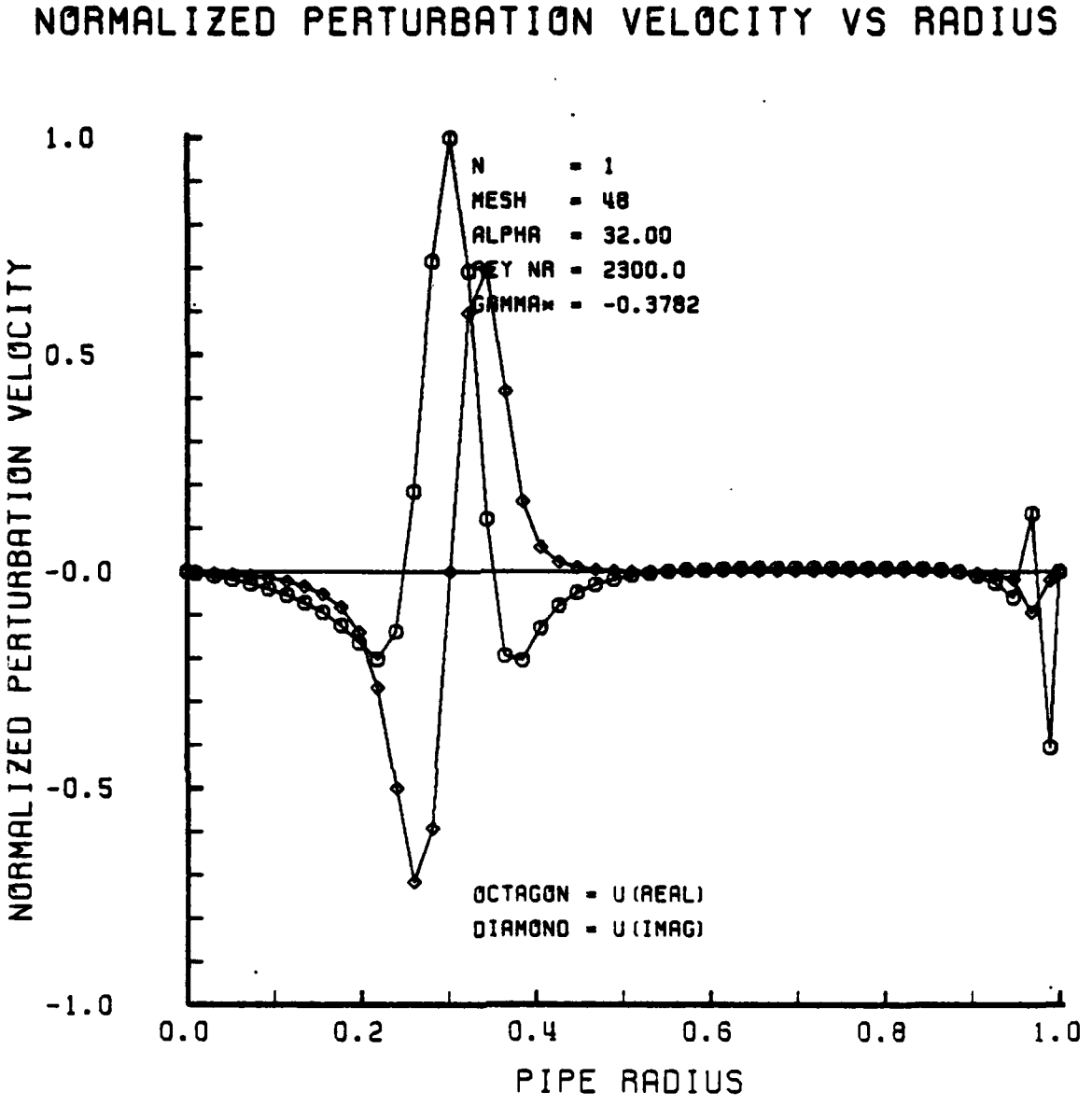

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 32.00
REY NR = 2300.0
GAMMA* = -0.3782
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-21

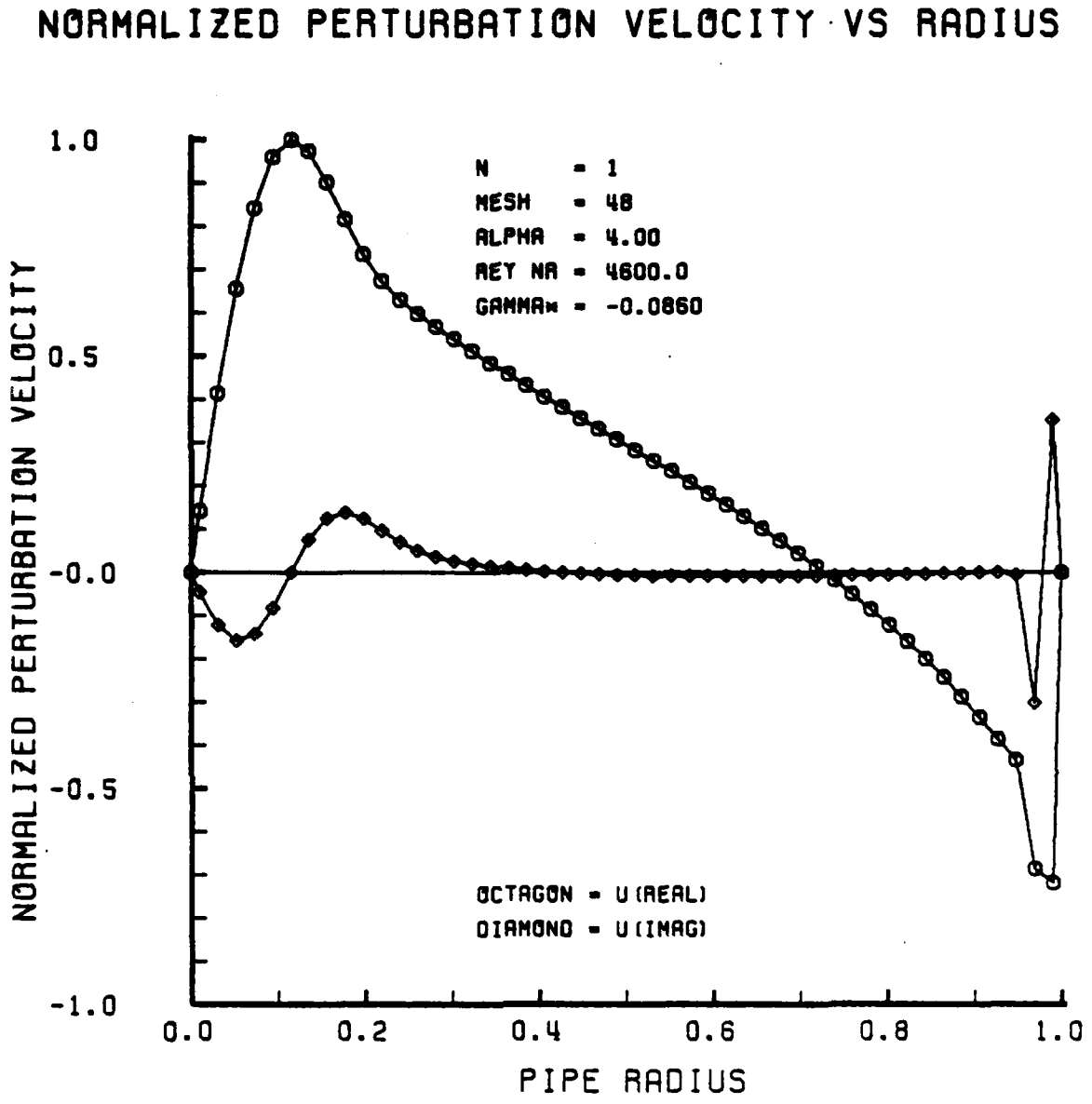

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 4.00
REY NR = 4600.0
GAMMA= -0.0860
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-22

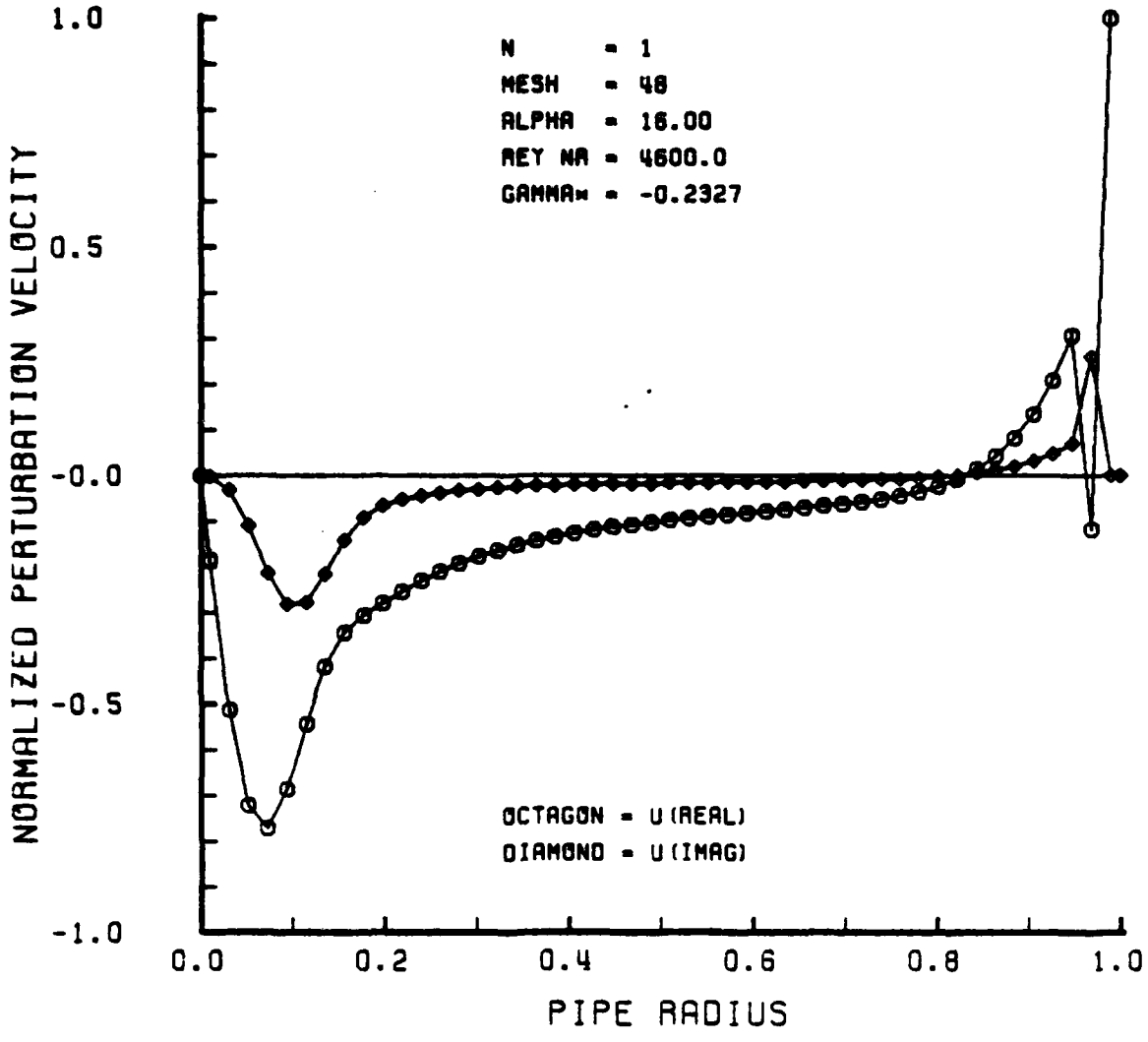

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 16.00
REY NR = 4600.0
GAMMA= = -0.2327
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-23

C. RESULTS FOR n = 6

Results for angular wave number n = 6 are similar to those for n = 0 in that the flow is stable at all Reynolds numbers. Stability data is presented in Tables 6-4 and 6-5 where it can be determined that this is the case. The data is once again presented as Reynolds number contours in the plots of GAMMA* vs. alpha in Figures 6-24 and 6-25 and as alpha contours in the plot of GAMMA* vs. Reynolds number in Figure 6-26. The trends are clear; the flow is stable at all Reynolds numbers, increasing Reynolds number has a destabilizing effect on the flow and increasing the axial wave number has a stabilizing effect.

Figures 6-27 through 6-36 are representative perturbation velocity plots for n = 6 at several Reynolds numbers and axial wave numbers. It can be seen in Figures 6-31 and 6-32 that the perturbation activity shifts from the wall to the axis for increasing α.

Table 6-4. Stability Data for Angular Wave Number n = 6

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=1.0	α=2.0	α=4.0	α=8.0	α=16.0	α=32.0
300	-0.3291	-0.3512	-0.4829	-0.7295	-1.1654	-1.9718	-2.8067	-4.6185
575	-0.1717	-0.2450	-0.3644	-0.5651	-0.9064	-1.4798	-2.0927	-
1150	-0.0858	-0.1814	-0.2789	-0.4386	-0.7049	-1.0532	-	-
2300	-0.0429	-0.1390	-0.2173	-0.3447	-0.5136	-	-	-
4600	-0.0215	-0.1084	-0.1712	-0.2549	-0.3631	-	-	-
9200	-0.0107	-0.0855	-0.1272	-0.1802	-0.2565	-	-	-
18400	-0.0054	-0.0636	-0.0899	-0.1273	-0.1811	-	-	-

Note: Values are based on 50 mesh points.

Table 6-5. Stability Data for Angular Wave Number n = 6

Growth or Decay Trends of Least Stable Eigenvalue, γ^*

Re	α=0.0	α=0.05	α=0.10	α=0.15	α=0.20	α=0.50	α=0.75	α=1.0
300	-0.3291	-0.3267	-0.3228	-0.3201	-0.3192	-0.3512	-0.4156	-0.4829
575	-0.1717	-0.1686	-0.1665	-0.1679	-0.1727	-0.2450	-0.3071	-0.3644
1150	-0.0858	-0.0833	-0.0864	-0.0960	-0.1092	-0.1814	-0.2325	-0.2789
2300	-0.0429	-0.0432	-0.0546	-0.0675	-0.0794	-0.1390	-0.1801	-0.2173
4600	-0.0215	-0.0273	-0.0397	-0.0505	-0.0603	-0.1084	-0.1414	-0.1712
9200	-0.0107	-0.0198	-0.0302	-0.0390	-0.0469	-0.0855	-0.1102	-0.1272
18400	-0.0054	-0.0151	-0.0234	-0.0305	-0.0368	-0.0636	-0.0779	-0.0899

Note: Values are based on 50 mesh points.

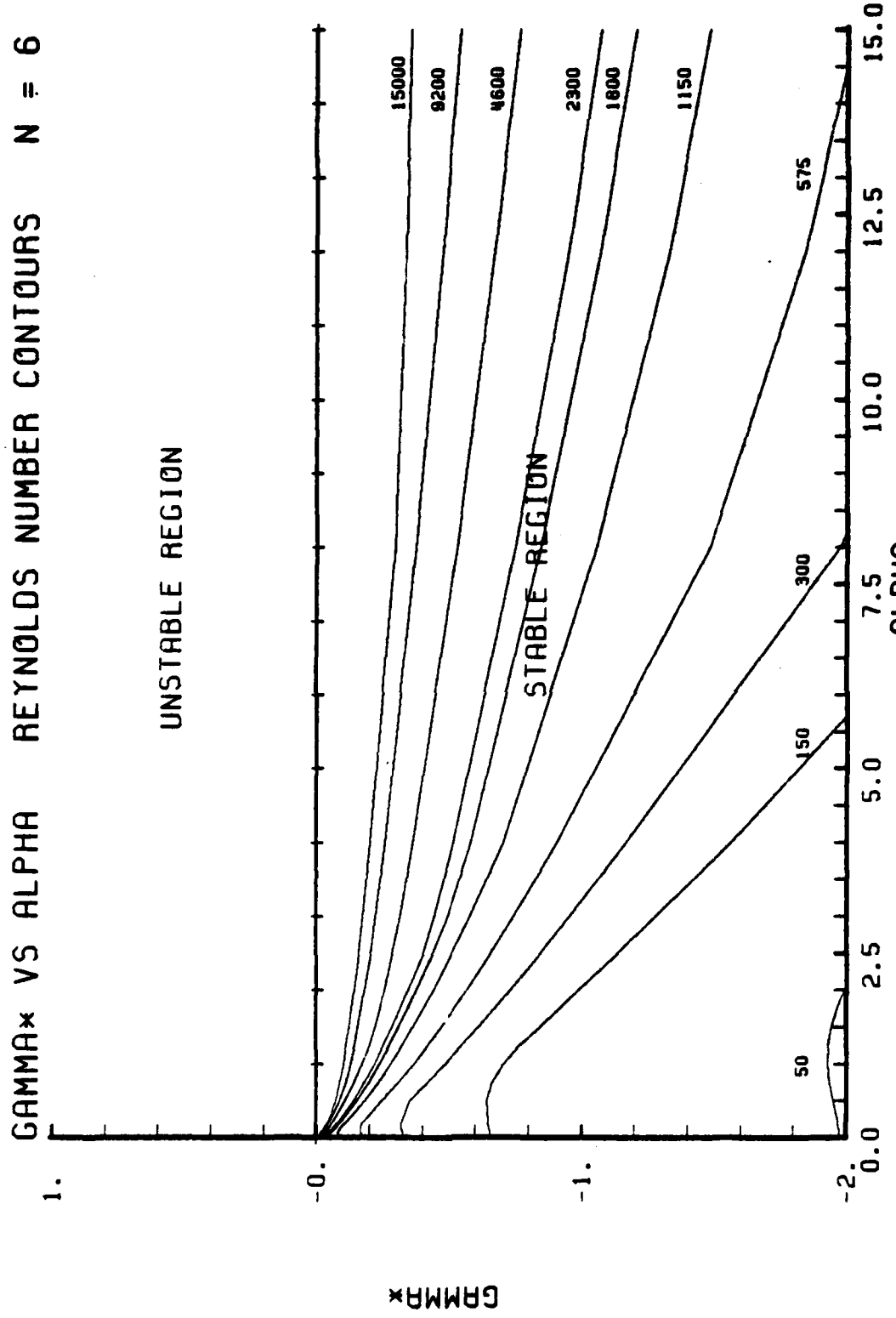

GAMMA* VS ALPHA REYNOLDS NUMBER CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
50
1.
-0.
-1.
-2.
0.0
2.5
5.0
7.5
10.0
12.5
15.0
ALPHA
GAMMA*

Figure 6-24

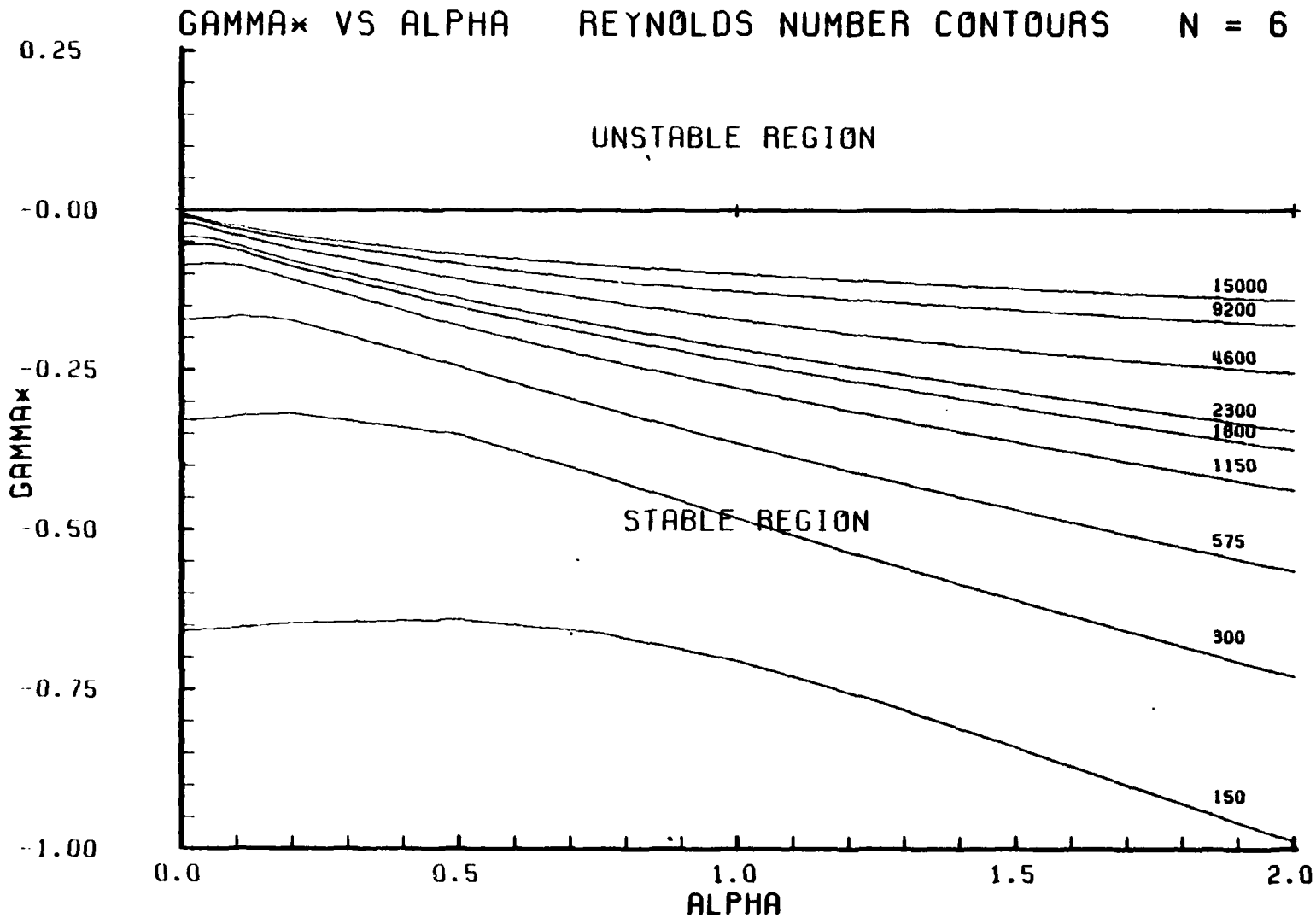

GAMMA× VS ALPHA REYNOLDS NUMBER CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
15000
9200
4600
2300
1800
1150
575
300
150
0.25
-0.00
-0.25
-0.50
-0.75
-1.00
GAMMA×
0.0
0.5
1.0
1.5
2.0
ALPHA

Figure 6-25

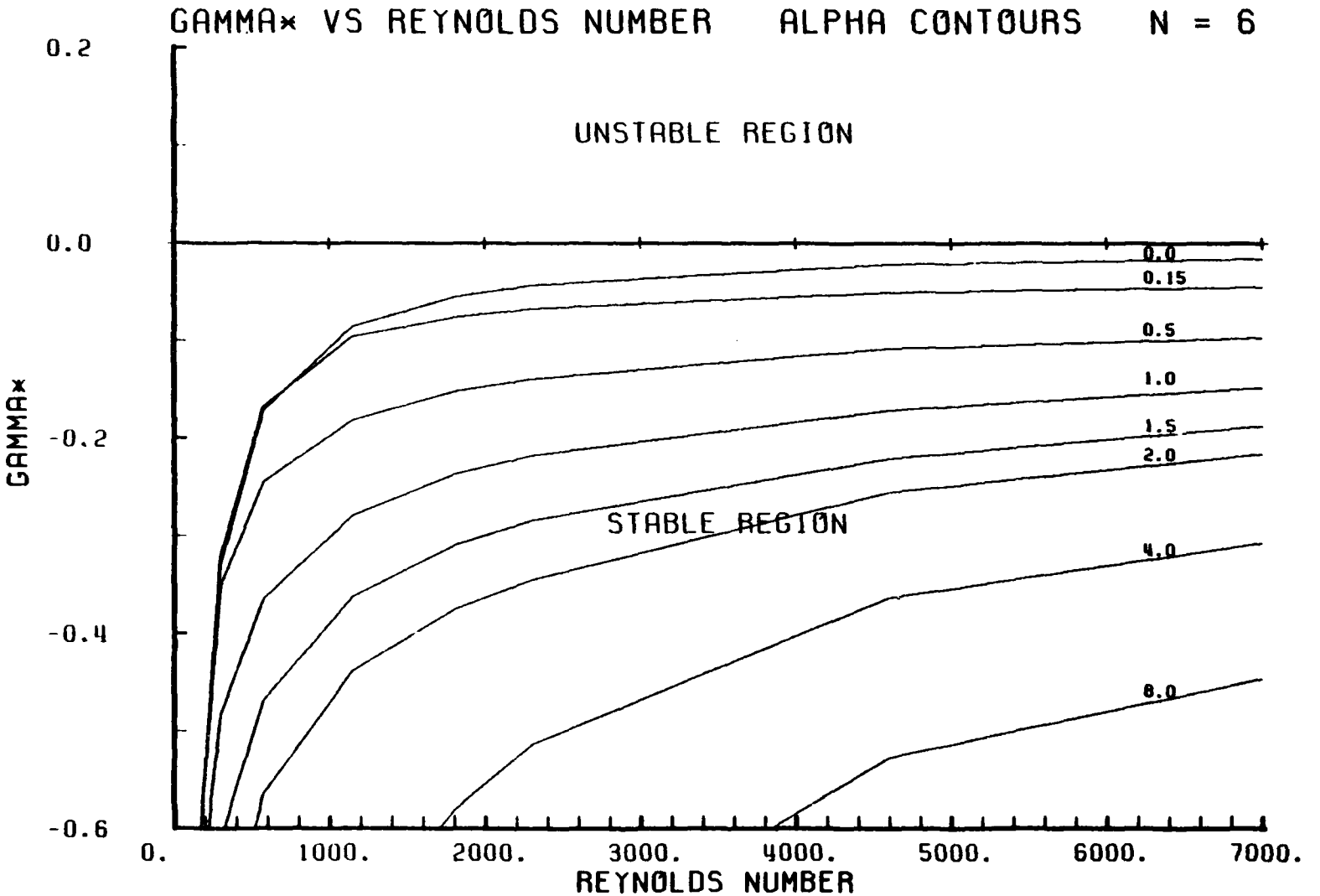

GAMMA* VS REYNOLDS NUMBER ALPHA CONTOURS N = 6
UNSTABLE REGION
STABLE REGION
0.0
0.15
0.5
1.0
1.5
2.0
4.0
8.0
0.2
0.0
-0.2
-0.4
-0.6
GAMMA*
0.
1000.
2000.
3000.
4000.
5000.
6000.
7000.
REYNOLDS NUMBER

Figure 6-26

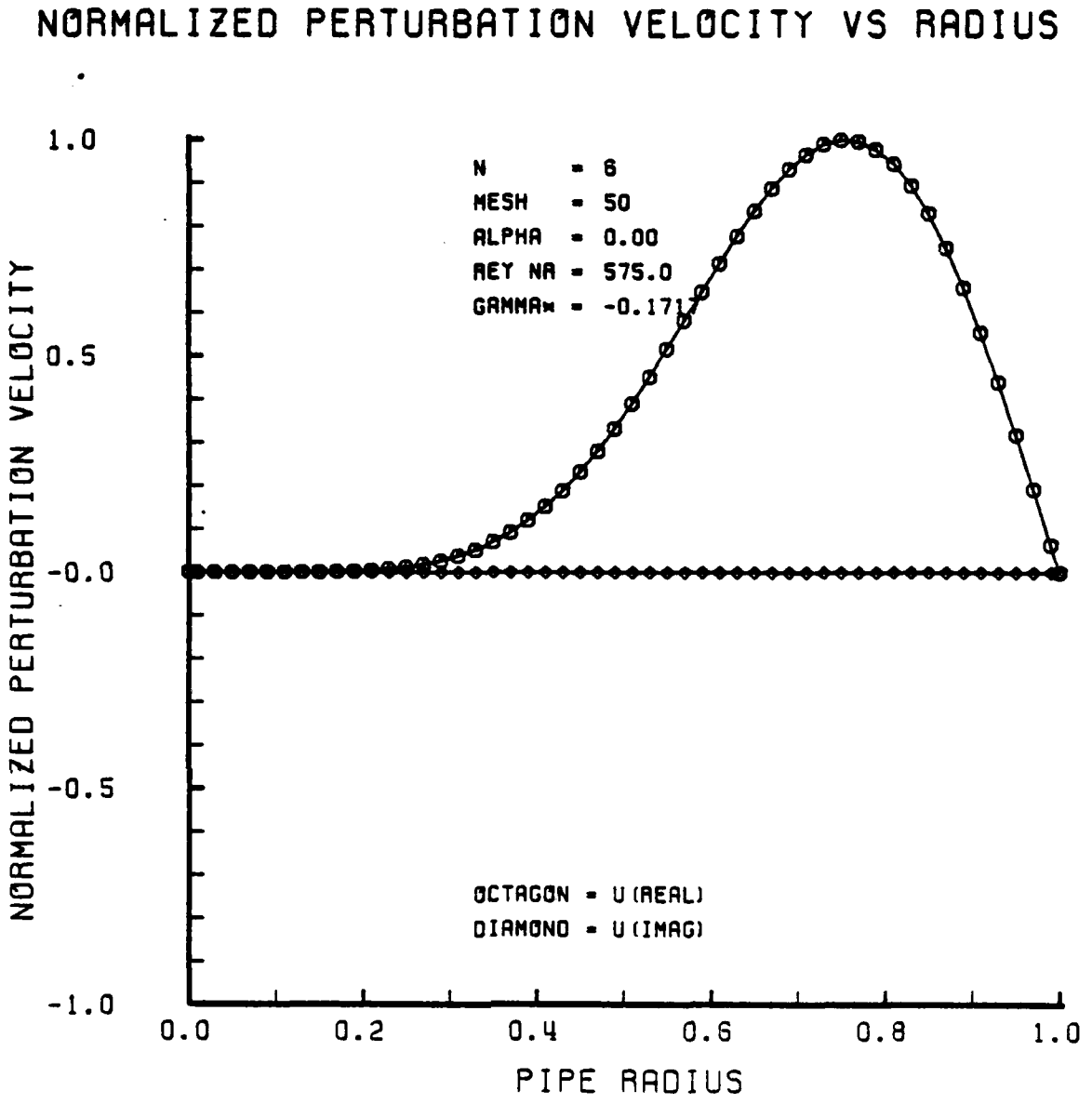

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.00
REY NR = 575.0
GAMMA* = -0.171
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-27

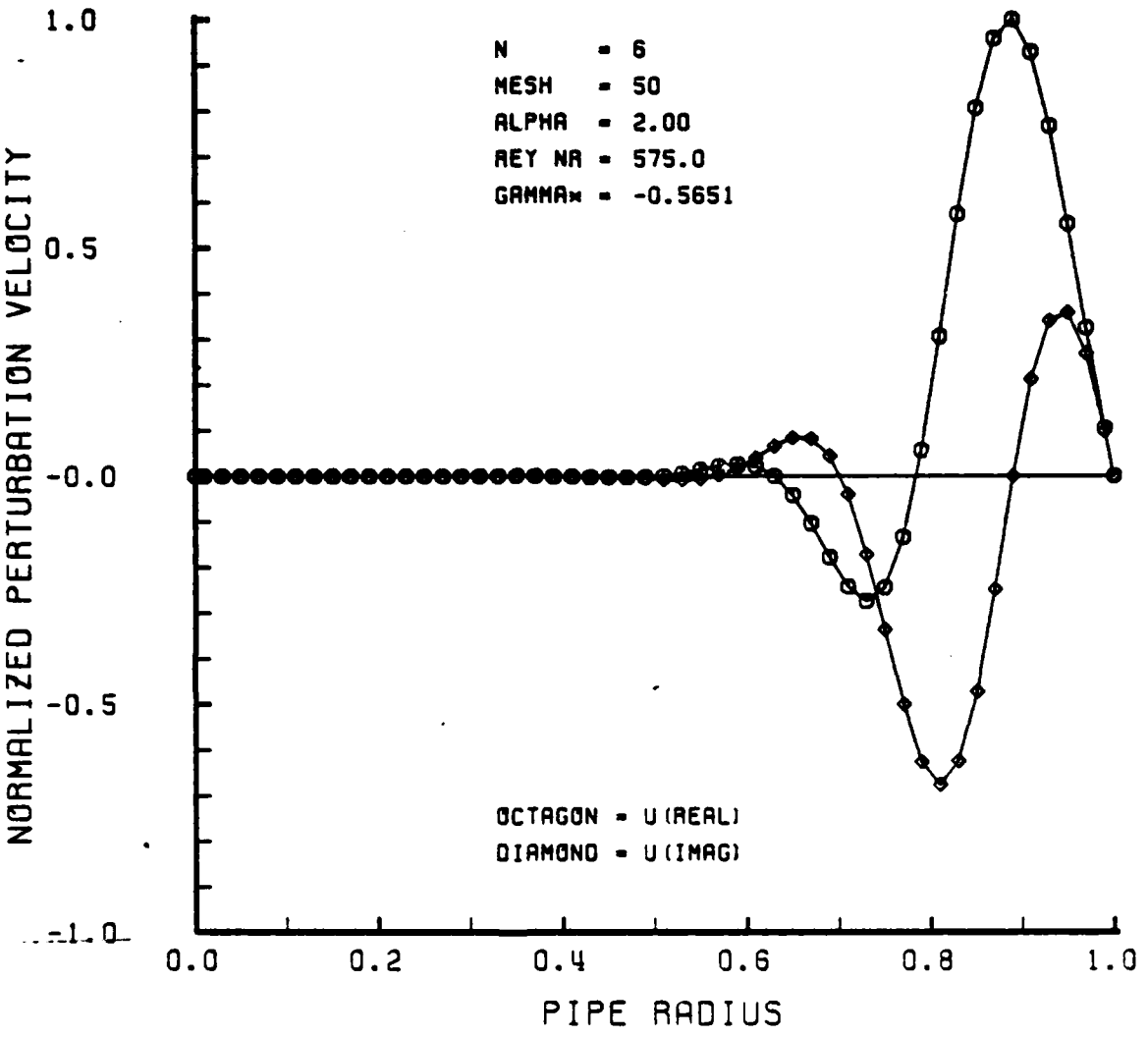

NORMALIZED PERTURBATION VELOCITY VS RADIUS.
N = 6
MESH = 50
ALPHA = 2.00
REY NR = 575.0
GAMMA* = -0.5651
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-28

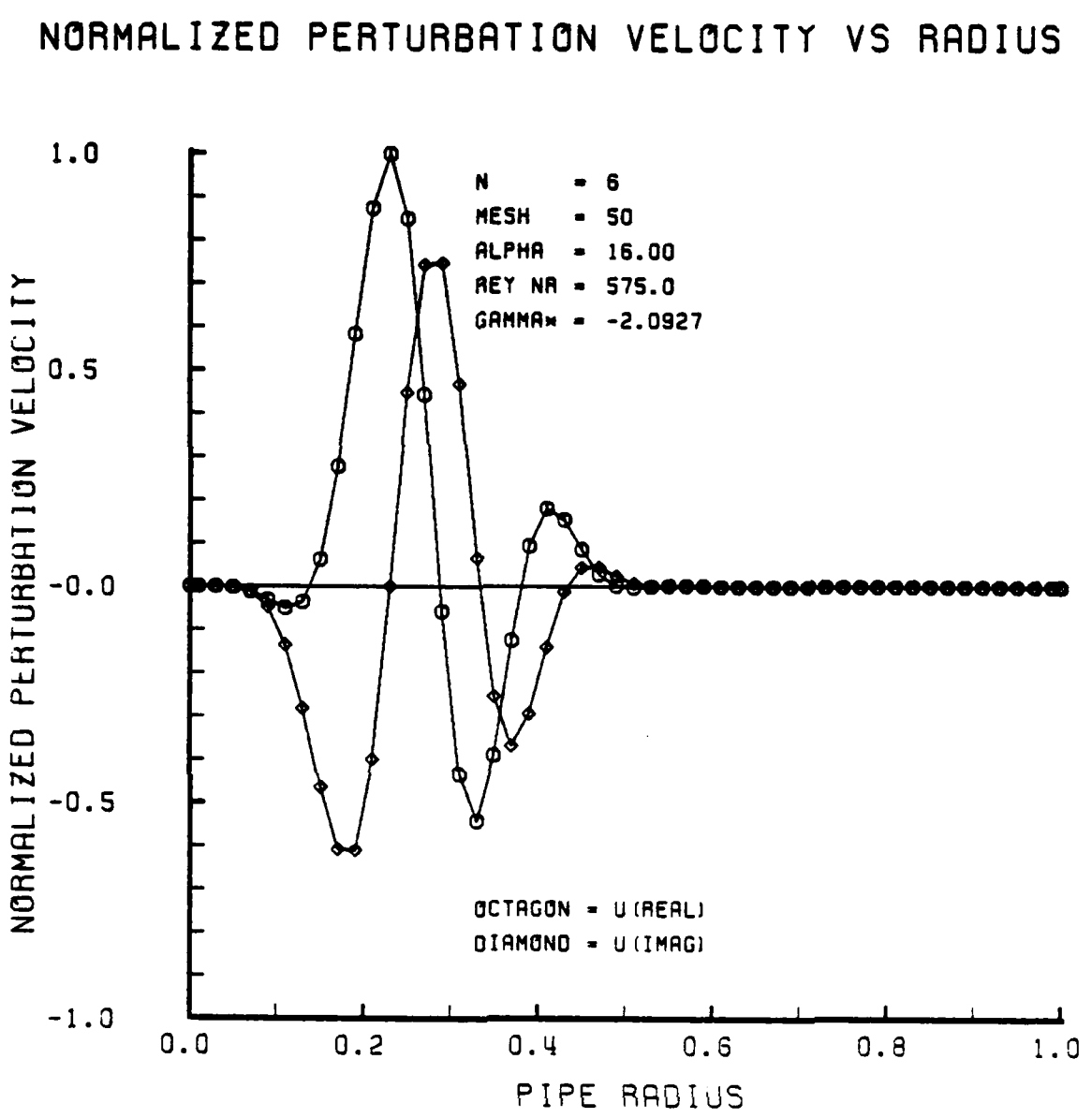

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 16.00
REY NR = 575.0
GAMMA* = -2.0927
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-29

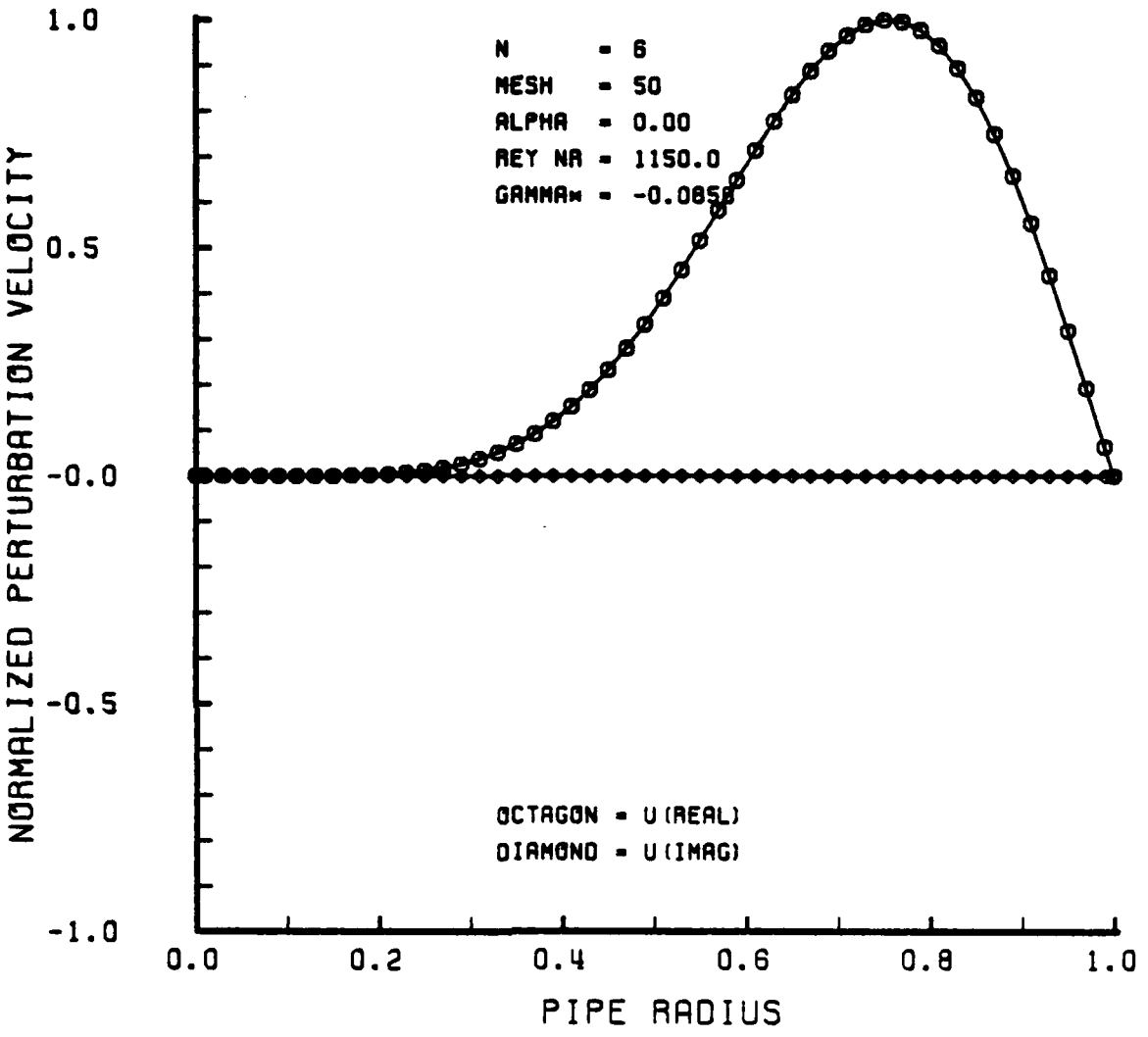

NORMALIZED PERTURBATION VELOCITY VS RADIUS
1.0
0.5
-0.0
-0.5
-1.0
NORMALIZED PERTURBATION VELOCITY
N = 6
MESH = 50
ALPHA = 0.00
REY NR = 1150.0
GAMMA = -0.0858
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-30

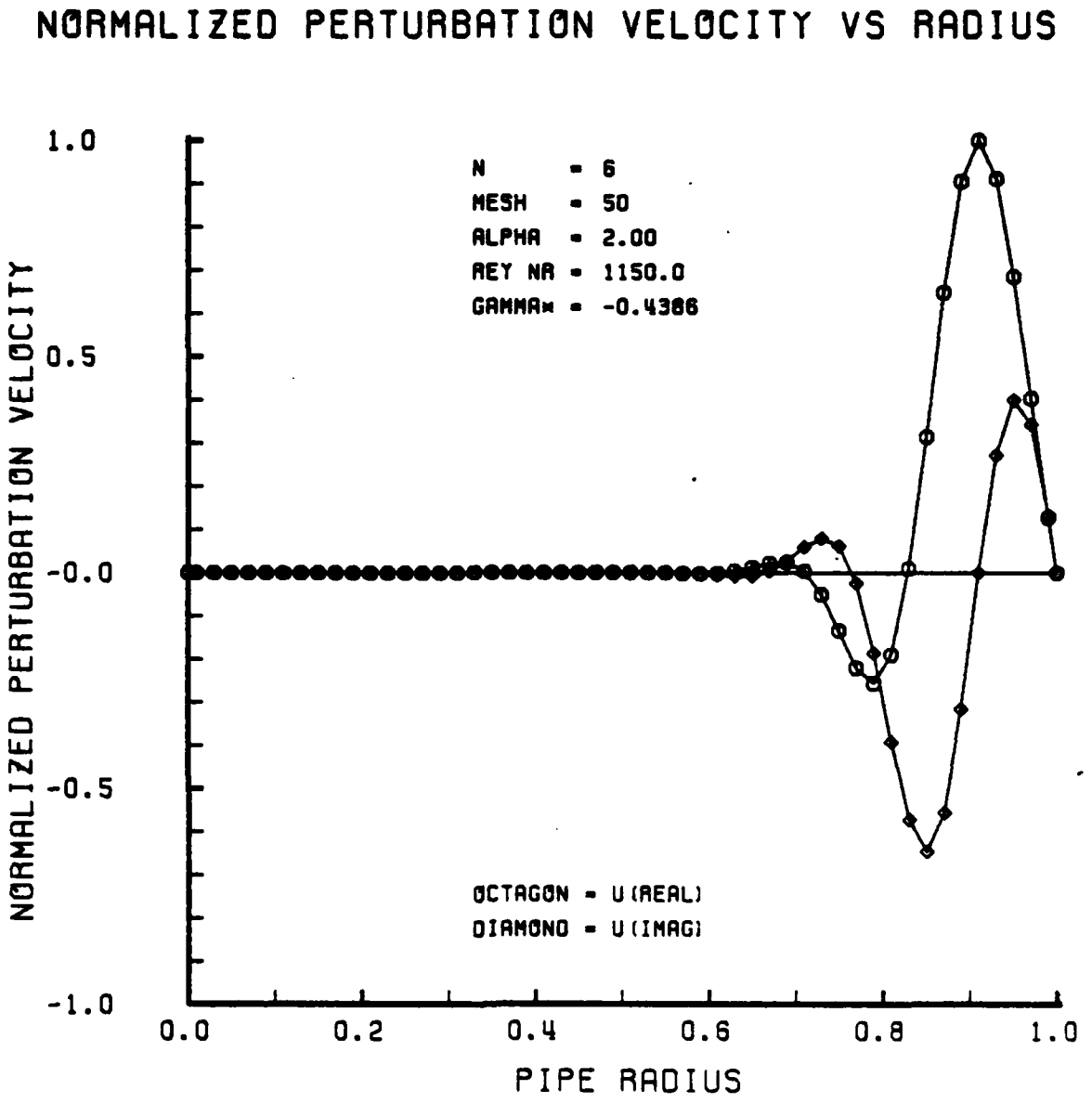

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 2.00
REY NR = 1150.0
GAMMA= = -0.4386
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-31

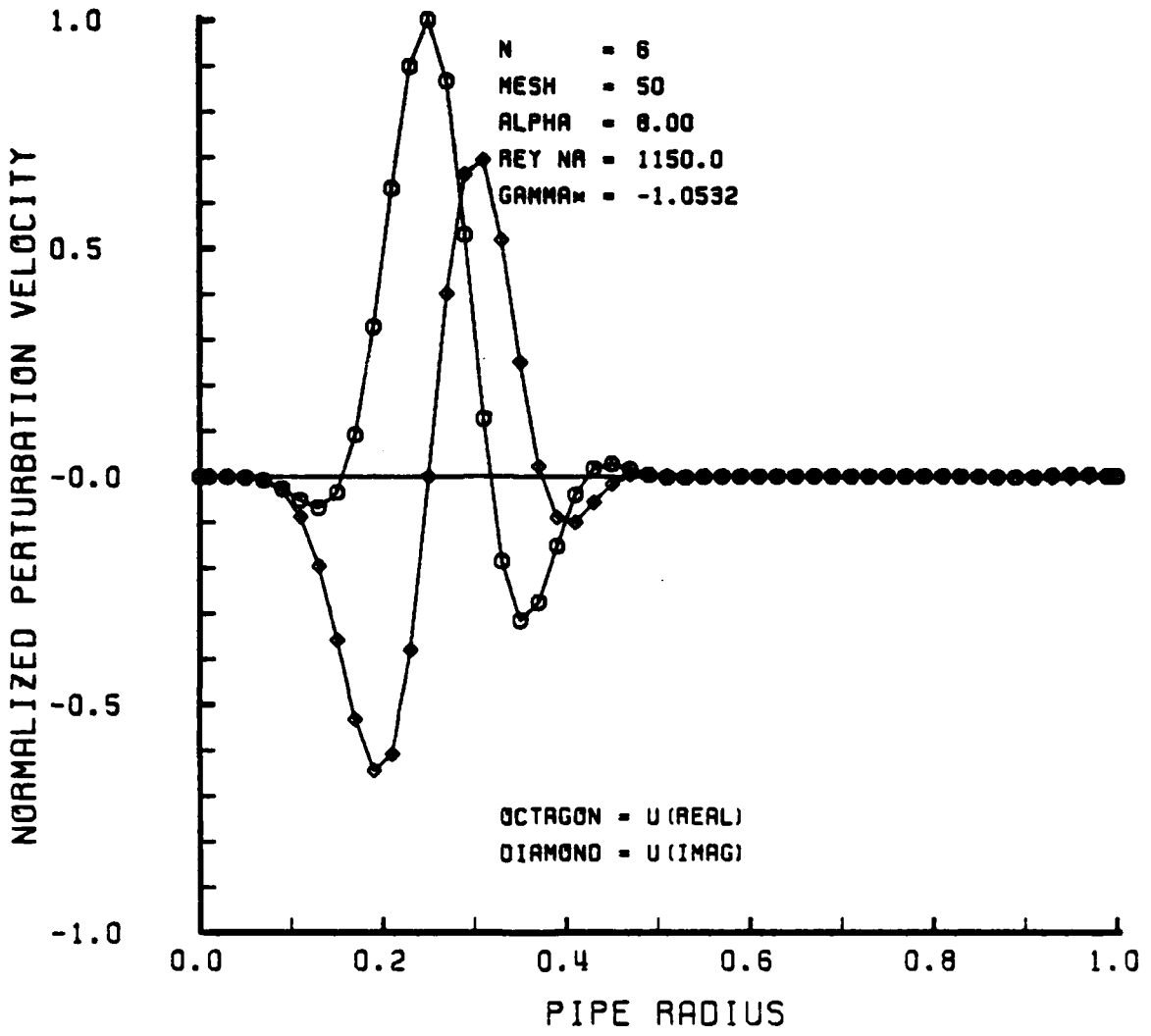

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.00
REY NR = 1150.0
GAMMA= = -1.0532
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-32

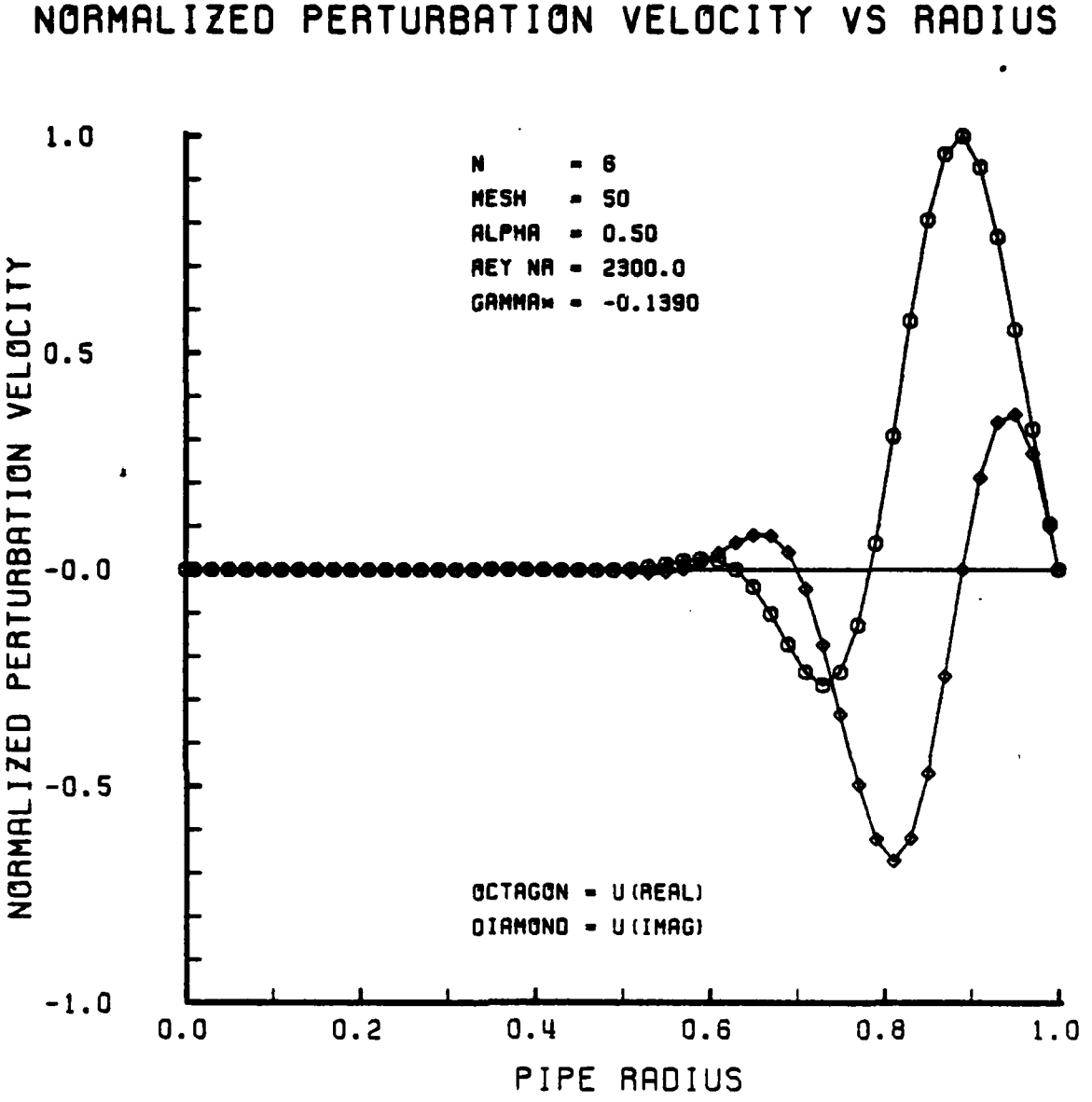

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 0.50
REY NR = 2300.0
GAMMA* = -0.1390
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-33

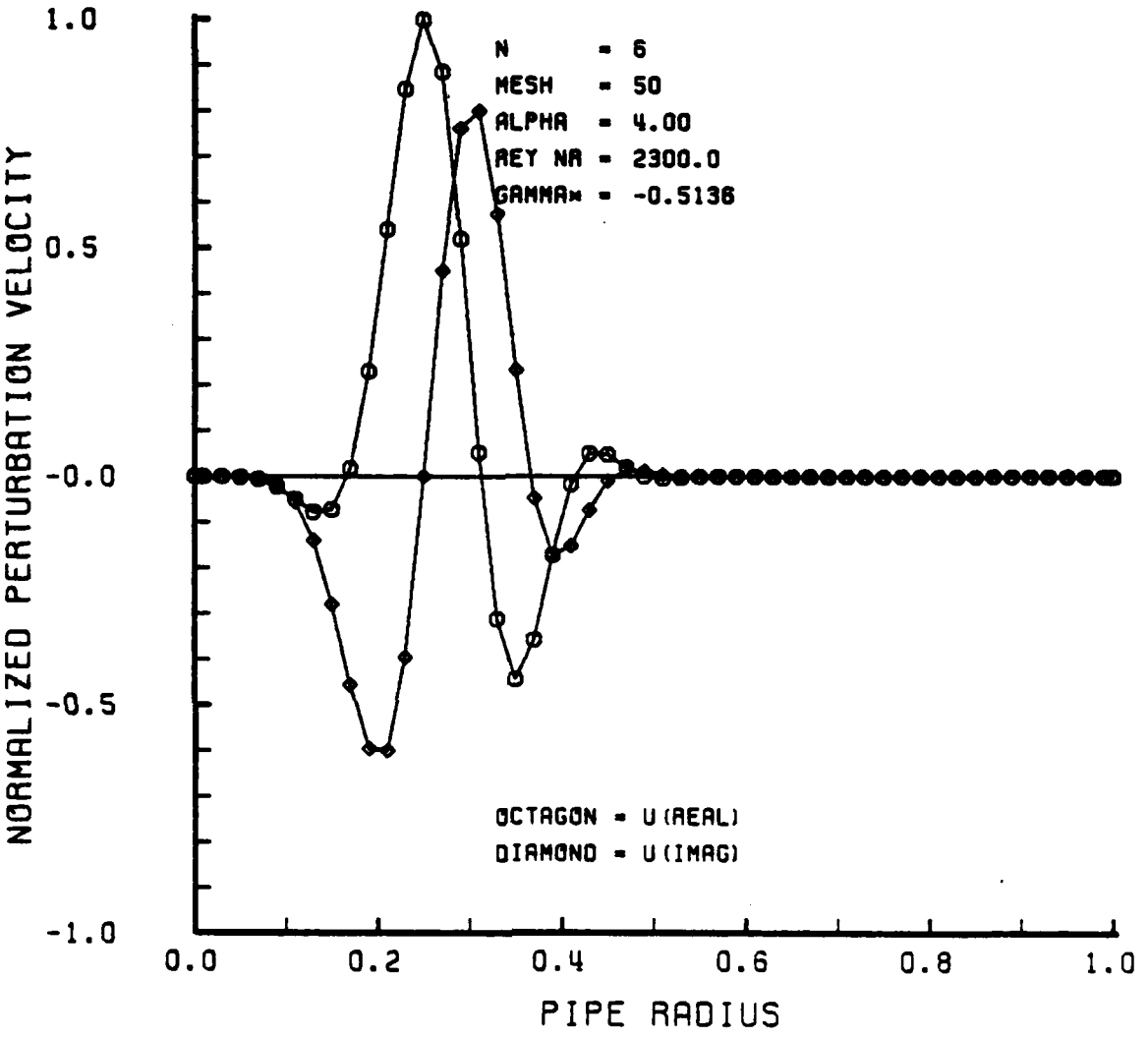

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 4.00
REY NR = 2300.0
GAMMA= -0.5136
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-34

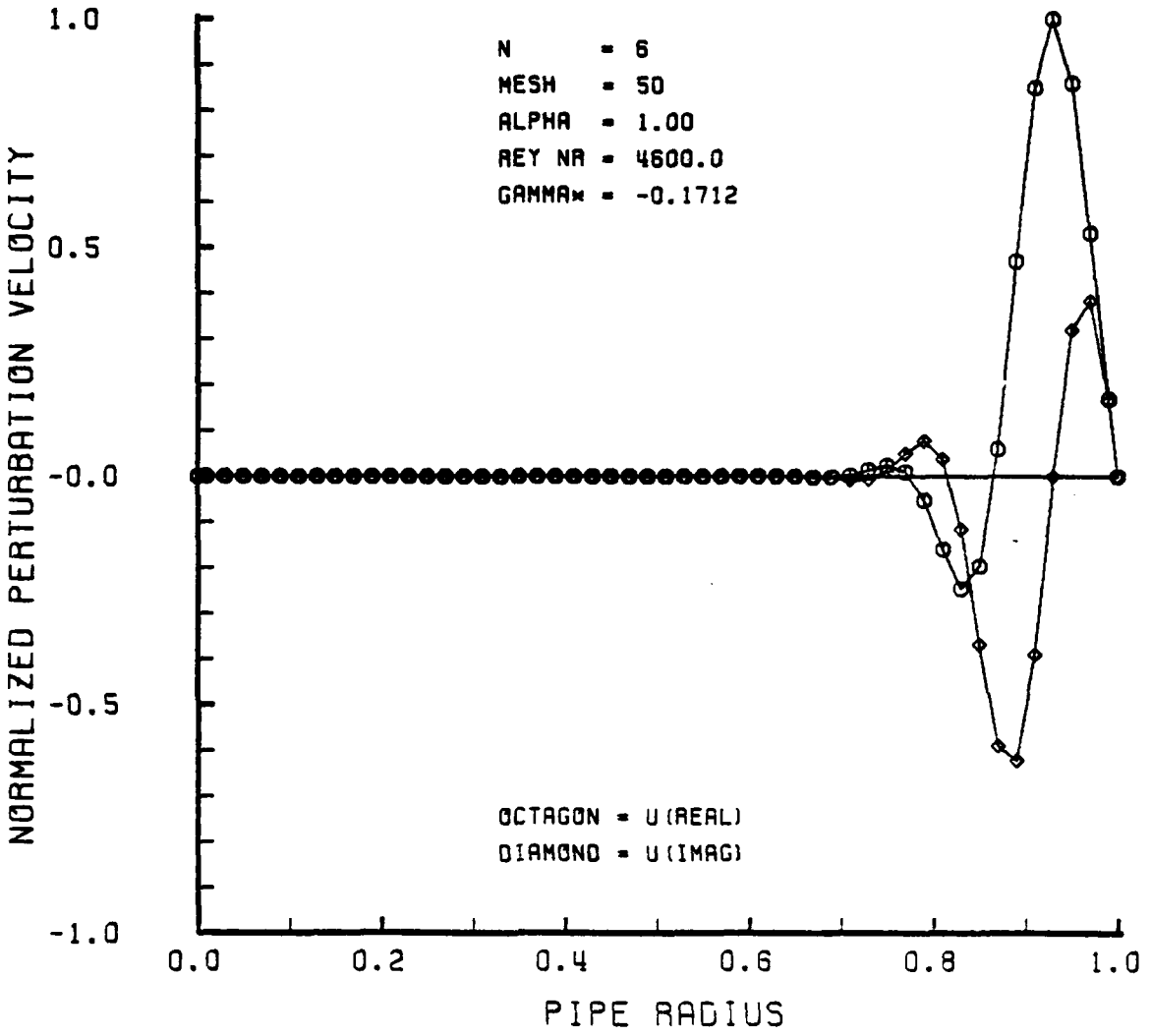

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 1.00
REY NR = 4600.0
GAMMA* = -0.1712
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-36

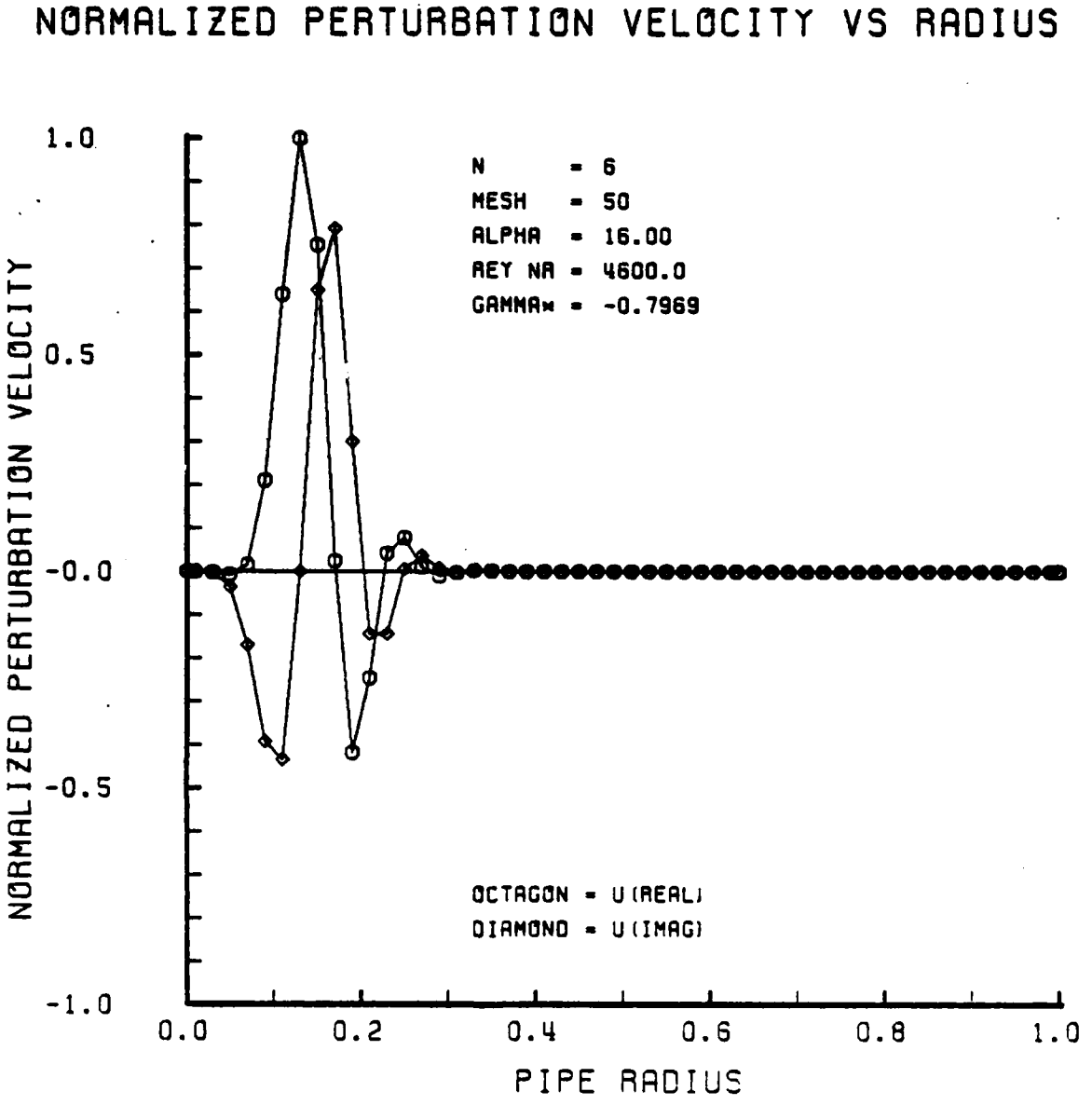

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 16.00
REY NR = 4600.0
GAMMAR = -0.7969
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-36

D. NUMERICAL ACCURACY

Numerical accuracy of the solutions of the governing vorticity transport equations was verified by substituting the least stable eigenvalue and its corresponding eigenvector into the eigensystem relation given in equation (5-39). The error vector was examined to determine the residual error of the solution. For the most part the error was of the order 10^{-12} to 10^{-9}, a most satisfactory result.

Upon closer examination of the error vector and the stability trends in the plots, it became evident that the accuracy of the numerical solution for the governing equations was not only dependent on the Reynolds number as previously mentioned but on the axial wave number as well. Plots of the perturbation velocity vs. radius for very large axial wave numbers are contained in Figures 6-37 through 6-39. The three figures, one for each angular wave number investigated, are examples of inaccurate solutions. Dependence of the solution on Reynolds number and axial wave number, became more evident as Re and α were increased. Rapid excursions of the perturbation velocity over a small portion of the pipe radius indicate that the computational mesh was not sufficiently fine to approximate the perturbation velocity function in the region of activity. This supposition can also be supported by examining the error vector where the residual error was found to be of the order 10^{-1} or worse, in the region of perturbation activity. Recall for $n = 1$ that some of the activity remained at the wall as α was increased. Refer to Figures 6-20 through 6-23. The residual error was

nominally of the order 10^{-5} in this region. Therefore, the true representation of the perturbation velocity is directly dependent upon an accurate numerical solution of the governing equations. These inaccuracies can be resolved by using a nonuniform mesh in the region of interest as suggested by Arnold. It is suspected, however, that after a thorough analysis of the effects of the problem parameters and mesh size on the solution have been investigated, only small values of the axial wave number below about 10.0 and Reynolds numbers in the vicinity of 1150 will be of primary interest. It is felt by this investigator that a mesh size of N = 50 and the wide range of problem parameters used here yielded solutions that satisfied the governing vorticity transport equations to a high degree and were sufficient to show stability trends.

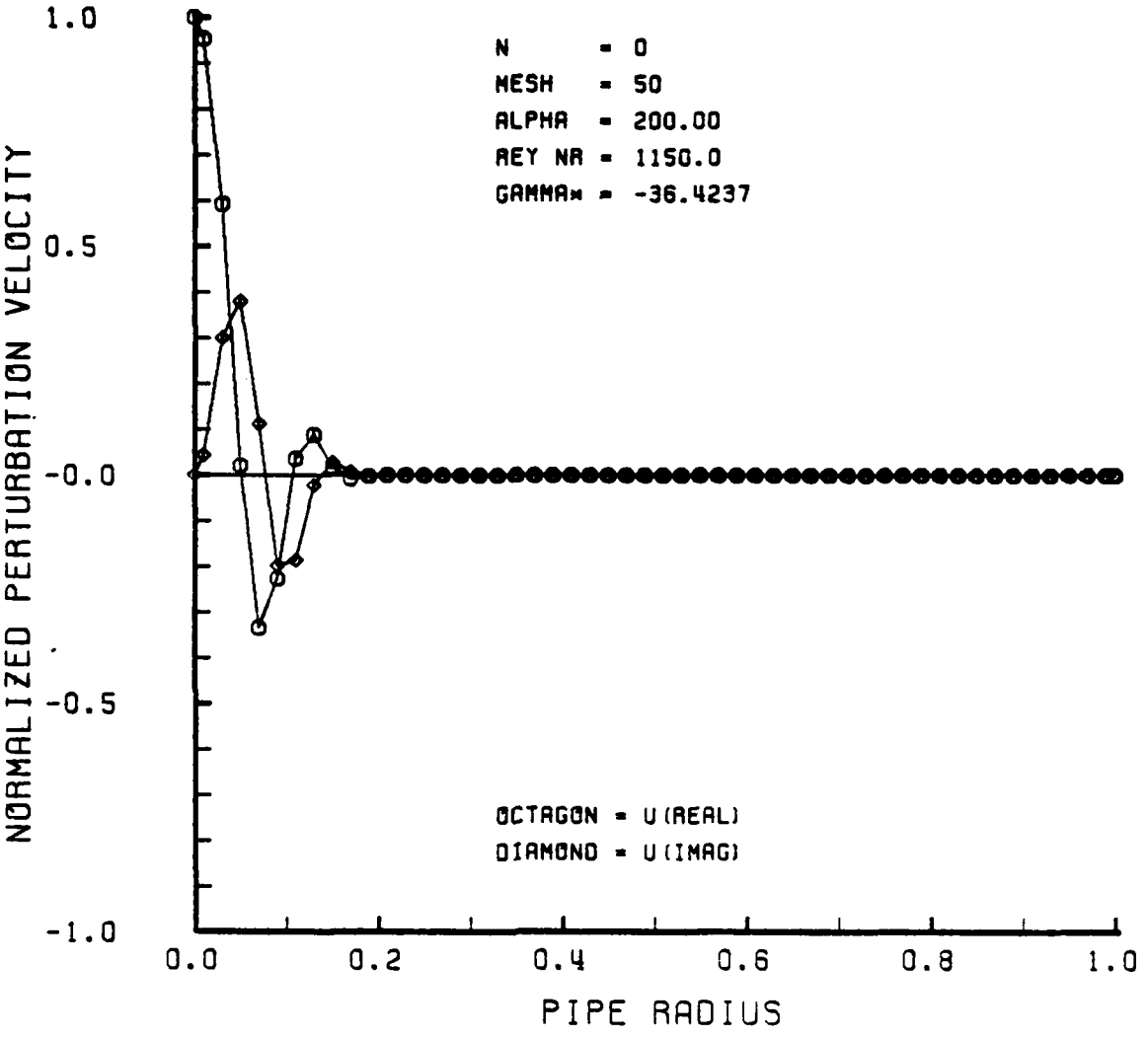

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 0
MESH = 50
ALPHA = 200.00
REY NR = 1150.0
GAMMA* = -36.4237
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-37

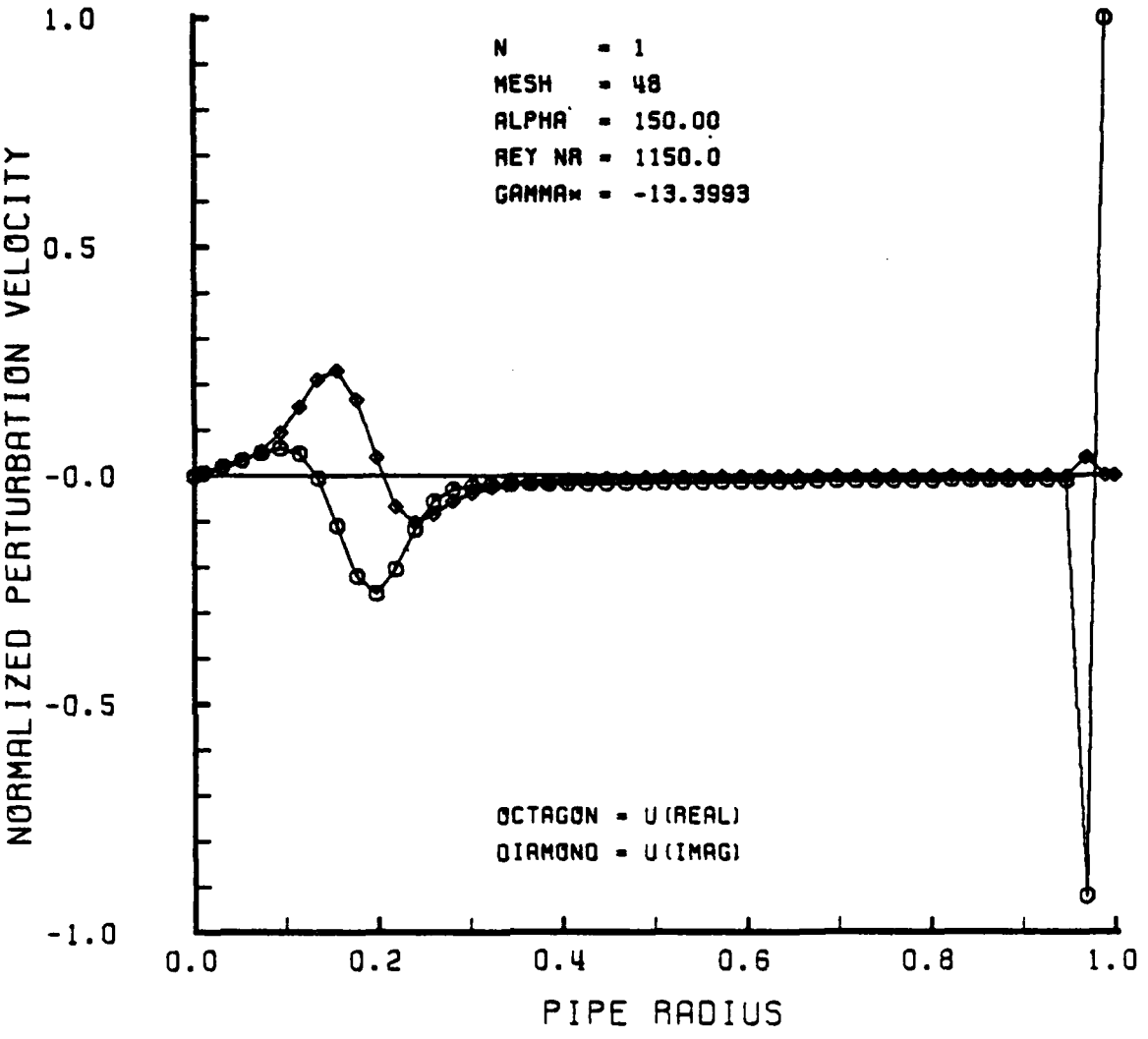

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 1
MESH = 48
ALPHA = 150.00
REY NR = 1150.0
GAMMA* = -13.3993
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-38

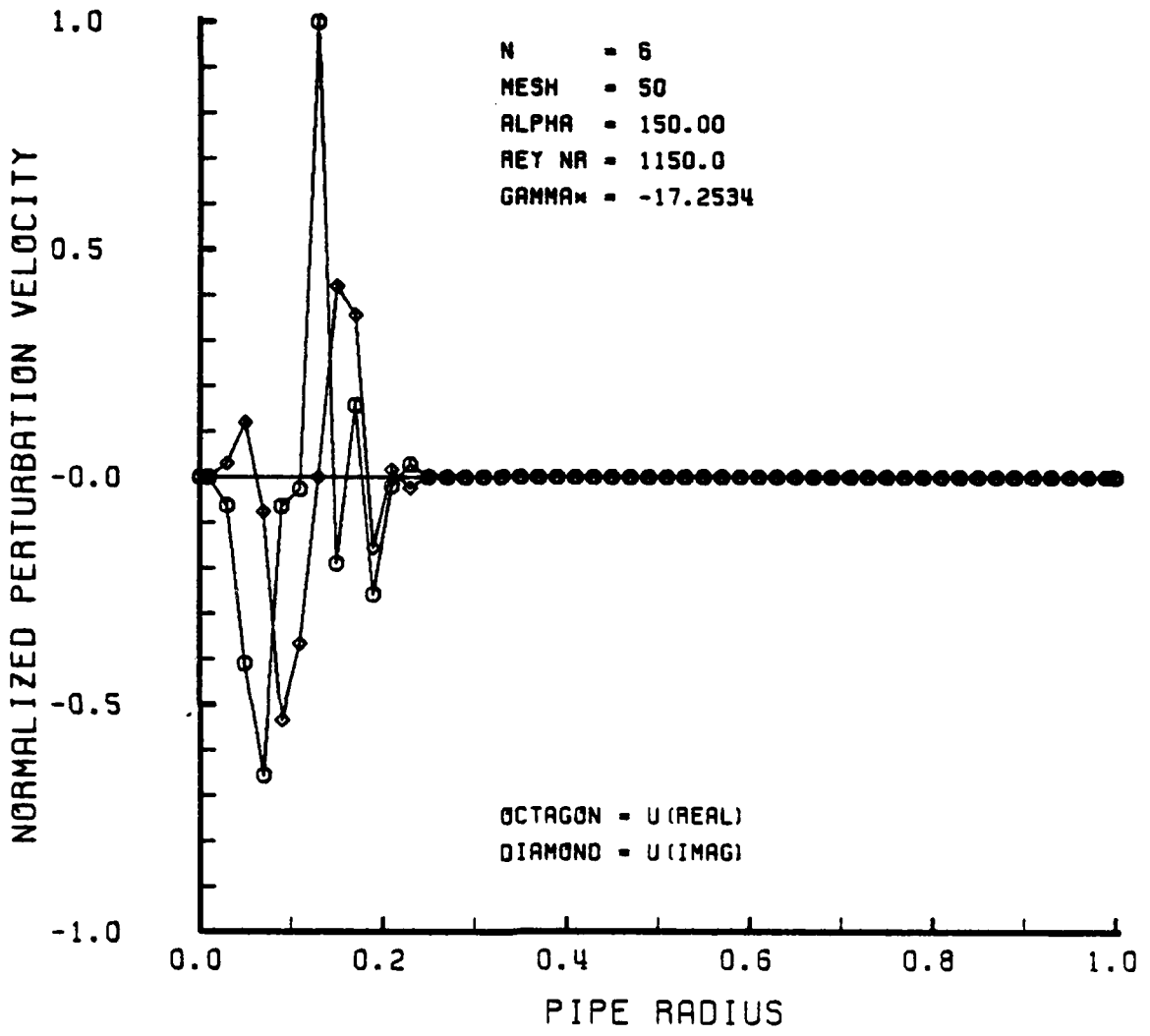

NORMALIZED PERTURBATION VELOCITY VS RADIUS
N = 6
MESH = 50
ALPHA = 150.00
REY NR = 1150.0
GAMMA* = -17.2534
OCTAGON = U(REAL)
DIAMOND = U(IMAG)
NORMALIZED PERTURBATION VELOCITY
1.0
0.5
-0.0
-0.5
-1.0
0.0
0.2
0.4
0.6
0.8
1.0
PIPE RADIUS

Figure 6-39

VII. RECOMMENDATIONS AND CONCLUSIONS

The problem of predicting the transition from laminar to turbulent flow of incompressible fluids in circular pipes has been pursued for nearly 100 years and has been presented here in a straightforward manner. The governing equations for the dynamics of this problem consist of the Navier-Stokes equation and the continuity equation. After taking the curl of the Navier-Stokes equation and applying the continuity principle, the linearized vorticity transport vector equation is obtained. The velocity vector potential function, which contains functions of r and a complex exponential factor, was then introduced. The resulting two linearily independent equations form the basis for the solution of the problem in the form of the vorticity transport matrix equation.

A valid solution of the governing equations was made possible because of two factors; (1) the advancements in the linearized theory by introducing a purely imaginary exponential form of the axial wave number, and (2) the rigorous method in which the boundary conditions at the pipe axis were derived and those constraints enforced. After an appropriate change of variables was performed, the governing equations were transformed into an equivalent pair of coupled, homogeneous differential equations that were solved in eigenvalue problem format. Accurate solutions of the governing equations were achieved by the refinement and combination of several standard methods of numerical analysis. These methods include; (1) the approximation of the governing equations along

a computational radial mesh developed by the half-station method, (2) the use of central and noncentral finite difference approximations having consistent fourth order truncation error, and (3) the use of complex, double precision number representations.

It was this author's intention from the outset to make this work as complete as possible. This was accomplished by including the development of the theory, the derivation of the boundary conditions, and a detailed explaination of the numerical methods used as well as obtaining solutions to the governing equations. This will enable future investigators to use this thesis as a starting point and as a single source reference for follow-on studies of this subject.

The conclusions of this numerical analysis follow directly from the results of the pipe flow stability problem which were discussed thoroughly in the previous chapter.

The most significant findings are:

(1) for n = 0, the flow is stable at all Re and α.

(2) for n = 1, the flow appears to be unstable at all Re for small values of α. This result is unfortunately in direct conflict with experimental observations.

(3) for n = 1, unstable eigenvalues are associated with perturbation activity at the wall.

(4) for n = 6, the flow is stable at all Re and α.

The following observations are also of interest:

(5) for n = 0, 1, and 6, increasing Reynolds number has a destabilizing effect on the flow.

(6) for n = 0, 1, and 6, increasing axial wave number has a stabilizing effect on the flow for fixed Re.

(7) for n = 1 and 6, increasing axial wave number causes a shift in the perturbation activity from the wall to the axis for fixed Re.

The results for angular wave numbers n = 1 and 6 are believed to be new. Conclusions (1), (5), (6), and (7) are in agreement with experimental results or previous theoretical investigations.

It is this author's opinion that the numerical analysis of the vorticity transport equations is now complete for angular wave numbers n = 0 and 6. It is felt, however, that the numerical results for angular wave number n = 1 warrant further study. The unresolved paradox is quite puzzeling in that for n = 1 the flow appears to be unstable at all Reynolds numbers. Continued numerical investigations for n = 1 and follow-on studies of other angular wave numbers should prove fruitful in establishing a theory which predicts the transition from laminar to turbulent flow of fluids in pipes.

APPENDIX A

COEFFICIENTS OF THE VORTICITY TRANSPORT EQUATIONS

The generalized coefficient matrices which appear in the basic vorticity transport equations (2-40) are defined below.

$$[M_4] = \frac{1}{Re}\begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \tag{A-1}$$

$$[M_3] = \frac{1}{Re}\begin{bmatrix} 0 & \frac{in}{r} \\ 0 & -\frac{2}{r} \end{bmatrix} \tag{A-2}$$

$$[M_2] = \frac{1}{Re}\begin{bmatrix} \left(\frac{n^2}{r^2} + \alpha^2\right) & \frac{2\,in}{r^2} \\ -\frac{4\,in}{r^2} & \left(\frac{2n^2+3}{r^2} + 2\alpha^2\right) \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & 2i\alpha(1-r^2) \end{bmatrix} \tag{A-3}$$

$$[M_1] = \frac{1}{Re}\begin{bmatrix} -\left(\frac{n^2}{r^3} - \frac{\alpha^2}{r}\right) & -in\left(\frac{n^2+1}{r^3} + \frac{\alpha^2}{r}\right) \\ \frac{4\,in}{r^3} & -\left(\frac{2n^2+3}{r^3} - \frac{2\alpha^2}{r}\right) \end{bmatrix} + \begin{bmatrix} 0 & 2n\alpha\left(\frac{1}{r}-r\right) \\ 0 & 2i\alpha\left(\frac{1}{r}-r\right) \end{bmatrix} \tag{A-4}$$

$$[M_0] = \frac{1}{Re}\begin{bmatrix} -\left(\frac{n^2(n^2-1)}{r^4} + \frac{(2n^2+1)\alpha^2}{r^2} + \alpha^4\right) & -in\left(\frac{n^2-1}{r^4} + \frac{3\alpha^2}{r^2}\right) \\ 4in\left(\frac{n^2-1}{r^4} + \frac{\alpha^2}{r^2}\right) & -\left(\frac{(n^2+3)(n^2-1)}{r^4} + \frac{2(n^2+1)\alpha^2}{r^2} + \alpha^4\right) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha\left(\frac{n^2}{r^2} - (n^2-\alpha^2) - \alpha^2 r^2\right) & 2n\alpha\left(\frac{1}{r^2} + 1\right) \\ -4n\alpha\left(\frac{1}{r^2} - 1\right) & -2i\alpha\left(\frac{n^2+1}{r^2} - (n^2+1-\alpha^2) - \alpha^2 r^2\right) \end{bmatrix} \qquad \text{(A-5)}$$

$$[N_2] = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{(A-6)}$$

$$[N_1] = \begin{bmatrix} 0 & \frac{in}{r} \\ 0 & -\frac{1}{r} \end{bmatrix} \qquad \text{(A-7)}$$

$$[N_0] = \begin{bmatrix} \left(\frac{n^2}{r^2} + \alpha^2\right) & \frac{in}{r^2} \\ -\frac{2in}{r^2} & \left(\frac{n^2+1}{r^2} + \alpha^2\right) \end{bmatrix} \qquad \text{(A-8)}$$

APPENDIX B

COEFFICIENTS OF THE TRANSFORMED VORTICITY TRANSPORT EQUATIONS FOR n = 0, 1, and 6

A. COEFFICIENTS FOR n = 0

Since the vorticity transport equations uncouple for n = 0, only the solution for the eigenfunction Q(r) is sought. Therefore the details of the complete coefficient matrices are not required. The coefficients which appear here represent the matrix element (2,2) and correspond to the primed quantities of equation (2-43).

$$M_4' = - \frac{r}{Re} \tag{B-1}$$

$$M_3' = - \frac{6}{Re} \tag{B-2}$$

$$M_2' = \frac{1}{Re} \left(- \frac{3}{r} + 2\alpha^2 r \right) + 2i\alpha \, (r - r^3) \tag{B-3}$$

$$M_1' = \frac{1}{Re} \left(\frac{3}{r^2} + 6\alpha^2 \right) + 6i\alpha \, (1 - r^2) \tag{B-4}$$

$$M_0' = - \frac{\alpha^4 r}{Re} + 2i\alpha^3 \, (r^3 - r) \tag{B-5}$$

$$N_2' = - r \tag{B-6}$$

$$N_1' = - 3 \tag{B-7}$$

$$N_0' = \alpha^2 r \tag{B-8}$$

These coefficients also appear and are computed in part III of the main investigative computer program for n = 0.

B. COEFFICIENTS for n = 1

Since the vorticity transport equations do not uncouple for n = 1, the eigenfunctions P(r) and Q(r) must be solved for simultaneously. With the introduction of the additional parameter as a result of the change of variables, H(0) appears explicitly as an unknown in the system of equations. This requires the addition of two coefficient matrices to equation (2-40). The 2 X 1 column matrix $[M_5']$ appears on the left side of equation (2-47) and one 2 X 1 column matrix $[N_3']$ appears on the right side of equation (2-47). The complete coefficient matrices for n = 1 are defined below.

$$[M_4'] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{r^2}{Re} \end{bmatrix} \tag{B-9}$$

$$[M_3'] = \begin{bmatrix} 0 & \frac{ir}{Re} \\ 0 & -\frac{10r}{Re} \end{bmatrix} \tag{B-10}$$

$$[M_2'] = \begin{bmatrix} \frac{1}{Re}(1 + \alpha^2 r^2) & \frac{8i}{Re} \\ -\frac{4i}{Re} & \frac{1}{Re}(2\alpha^2 r^2 - 19) + 2i\alpha(r^2 - r^4) \end{bmatrix} \tag{B-11}$$

$$[M_1'] = \begin{bmatrix} \frac{1}{Re}\left(\frac{3}{r} + 5\alpha^2 r\right) & 2\alpha(r - r^3) \\ 0 & \frac{1}{Re}\left(\frac{3}{r} + 10\alpha^2 r\right) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{i}{Re}\left(\frac{12}{r} - \alpha^2 r\right) \\ -\frac{12i}{Rer} & 10i\alpha\,(r - r^3) \end{bmatrix} \qquad \text{(B-12)}$$

$$[M_0'] = \begin{bmatrix} \frac{\alpha^2}{Re}\,(1-\alpha^2 r^2) & 2\alpha(3 - r^2) \\ -4\alpha(1 - r^2) & \frac{1}{Re}\,(4\alpha^2 - \alpha^4 r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha(1 - r^2 + \alpha^2 r^2 - \alpha^2 r^4) & \frac{-5i\alpha^2}{Re} \\ \frac{4i\alpha^2}{Re} & 2i\alpha\{2(1 - r^2) - \alpha^2(r^2 - r^4)\} \end{bmatrix} \qquad \text{(B-13)}$$

$$[M_5'] = \begin{bmatrix} 2\alpha(2 - \alpha^2 + \alpha^2 r^2) + \frac{i\alpha^4}{Re} \\ -\frac{\alpha^4}{Re} - 2i\alpha^3\,(1 - \alpha r^2) \end{bmatrix} \qquad \text{(B-14)}$$

$$[N_2'] = \begin{bmatrix} 0 & 0 \\ 0 & -r^2 \end{bmatrix} \qquad \text{(B-15)}$$

$$[N_1'] = \begin{bmatrix} 0 & ir \\ 0 & -5r \end{bmatrix} \qquad \text{(B-16)}$$

$$[N_0'] = \begin{bmatrix} 1 + \alpha^2 r^2 & 3i \\ -2i & -2 + \alpha^2 r^2 \end{bmatrix} \qquad \text{(B-17)}$$

$$[N_3'] = \begin{bmatrix} -i\alpha^2 \\ \alpha^2 \end{bmatrix} \qquad \text{(B-18)}$$

These coefficients also appear and are computed in Part III of the main investigative computer program for n = 1.

C. COEFFICIENTS for n = 6

The vorticity transport equations for n = 6 remain coupled. There are no additional parameters introduced with the change of variables to P(r) and Q(r). The system of equations take the form of equation (2-40) once again. The complete coefficient matrices for n = 6 are defined below.

$$[M_4'] = \begin{bmatrix} 0 & 0 \\ 0 & -\frac{r^3}{Re} \end{bmatrix} \tag{B-19}$$

$$[M_3'] = \begin{bmatrix} 0 & \frac{6ir^2}{Re} \\ 0 & -\frac{14r^2}{Re} \end{bmatrix} \tag{B-20}$$

$$[M_2'] = \begin{bmatrix} \frac{r^2}{Re}(36 + \alpha^2 r^2) & 0 \\ 0 & \frac{r}{Re}(21 + 2\alpha^2 r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & \frac{66ir}{Re} \\ -\frac{24ir^2}{Re} & 2i\alpha(r^3 - r^5) \end{bmatrix} \tag{B-21}$$

$$[M_1'] = \begin{bmatrix} \frac{1}{Re}(252r + 9\alpha^2r^2) & 12\alpha(r^2 - r^4) \\ 0 & \frac{1}{Re}(315 + 14\alpha^2r^2) \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & -\frac{i}{Re}(42 + 6\alpha^2r^2) \\ -\frac{168ir}{Re} & 14i\alpha(r^2 - r^4) \end{bmatrix} \qquad \text{(B-22)}$$

$$[M_0'] = \begin{bmatrix} -\frac{1}{Re}(972 + 57\alpha^2r^2 + \alpha^4r^4) & 24\alpha(2r - r^3) \\ -24\alpha(r^2 - r^4) & -\frac{1}{Re}\left(\frac{1152}{r} + 56\alpha^2r + \alpha^4r^3\right) \end{bmatrix}$$

$$+ \begin{bmatrix} -2i\alpha r^2\{36-36r^2 + \alpha^2r^2(1-r^2)\} & -\frac{i}{Re}\left(\frac{768}{r} + 36\alpha^2r\right) \\ \frac{i}{Re}(648 + 24\alpha^2r^2) & -56i\alpha(r-r^3)-2i\alpha^3(r^3-r^5) \end{bmatrix} \qquad \text{(B-23)}$$

$$[N_2'] = \begin{bmatrix} 0 & 0 \\ 0 & -r^3 \end{bmatrix} \qquad \text{(B-24)}$$

$$[N_i'] = \begin{bmatrix} 0 & 6ir^2 \\ 0 & -7r^2 \end{bmatrix} \qquad \text{(B-25)}$$

$$[N_o'] = \begin{bmatrix} 36r^2 + \alpha^2 r^4 & 24ir \\ -12ir^2 & 28r + \alpha^2 r^3 \end{bmatrix} \qquad \text{(B-26)}$$

These coefficients also appear and are computed in part III of the main investigative computer program for n = 6.

APPENDIX C

COEFFICIENTS OF THE BOUNDARY EQUATIONS AT THE AXIS

The matrices which appear in the boundary equations at the axis, equations (3-4) through (3-8) are defined below.

$$[C_1] = \begin{bmatrix} -n^2 & -in \\ +4in & -(n^2+3) \end{bmatrix} \tag{C-1}$$

$$[C_2] = \begin{bmatrix} -n^2 & -2in \\ +4in & -(n^2+4) \end{bmatrix} \tag{C-2}$$

$$[C_3] = \begin{bmatrix} -\frac{n^2}{2} & -\frac{3}{2}in \\ +2in & -\frac{1}{2}(n^2+3) \end{bmatrix} \tag{C-3}$$

$$[C_4] = \begin{bmatrix} -\left|2n^2 - (2n^2+1)\frac{i\alpha}{Re}\right| & -in\left(2 - \frac{3i\alpha}{Re}\right) \\ +4in\left(1 - \frac{i\alpha}{Re}\right) & -2(n^2+1)\left(1 - \frac{i\alpha}{Re}\right) \end{bmatrix} \tag{C-4}$$

$$[D_4] = \begin{bmatrix} + n^2 & + in \\ - 2\,in & + (n^2 + 1) \end{bmatrix} \quad . \tag{C-5}$$

$$[C_5] = \begin{bmatrix} - \frac{n}{6} & - \frac{2i}{3} \\ + \frac{2i}{3} & - \frac{n}{6} \end{bmatrix} \tag{C-6}$$

$$[C_6] = \begin{bmatrix} - n & - 2i \\ + 2i & - n \end{bmatrix} \tag{C-7}$$

$$[C_7] = \begin{bmatrix} - \frac{n^2}{24} & - \frac{5\,in}{24} \\ + \frac{in}{6} & - \frac{(n^2 - 5)}{24} \end{bmatrix} \tag{C-8}$$

$$[C_8] = \begin{bmatrix} - \left\{ n^2 - \left(n^2 - \frac{3}{2}\right) \frac{i\alpha}{Re} \right\} & - in \left(3 - \frac{5i\alpha}{2Re}\right) \\ + 2\,in \left(1 - \frac{i\alpha}{Re}\right) & - (n^2 - 3) \left(1 - \frac{i\alpha}{Re}\right) \end{bmatrix} \tag{C-9}$$

$$[D_8] = \begin{bmatrix} + \frac{n^2}{2} & + \frac{3\,in}{2} \\ - ir & + \frac{(n^2 - 3)}{2} \end{bmatrix} \tag{C-10}$$

$$[C_9] = \begin{bmatrix} \left(\frac{i\alpha^3}{Re} + 2n^2 - 2\alpha^2\right) & -2in \\ -4in & \left(\frac{i\alpha^3}{Re} + 2n^2 + 2 - 2\alpha^2\right) \end{bmatrix} \quad \text{(C-11)}$$

$$[D_9] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \quad \text{(C-12)}$$

Column matrices used in equation (5-36) for n = 1.

$$[C_9^*] = \begin{bmatrix} \left(\frac{\alpha^3}{Re} - 4i + 2i\alpha^2\right) \\ \left(-2\alpha^2 + \frac{i\alpha^3}{Re}\right) \end{bmatrix} \{H(0)\} \quad \text{(C-13)}$$

$$[D_9^*] = \begin{bmatrix} -i \\ 1 \end{bmatrix} \{H(0)\} \quad \text{(C-14)}$$

APPENDIX D

SPECIAL CONDITIONS AT THE AXIS

The determinates are formed from the corresponding matrices as defined in Appendix C. For angular wave numbers $n \geq 6$, all of the determinates become non-singular.

$$|C_1| = n^2 (n^2 - 1) \tag{D-1}$$

$$|C_2| = n^2 (n^2 - 4) \tag{D-2}$$

$$|C_3| = \frac{n^2}{4} (n^2 - 9) \tag{D-3}$$

$$|i\alpha[C_4] - \gamma[D_4]| = (n^2 - 1) \left\{ -2\alpha^2 \left(1 - \frac{i\alpha}{Re}\right) \left[2n^2 - (2n^2 - 1) \frac{i\alpha}{Re}\right] + \gamma i\alpha \left[4n^2 - (4n^2 - 1) \frac{i\alpha}{Re}\right] + \gamma^2 n^2 \right\} \tag{D-4}$$

$$|C_5| = \frac{1}{36} (n^2 - 16) \tag{D-5}$$

$$|C_6| = (n^2 - 4) \tag{D-6}$$

$$|C_7| = \frac{n^2}{576} (n^2 - 25) \tag{D-7}$$

$$|i\alpha[C_s] - \gamma[D_s]| = (n^2 - 9)\left\{-\alpha^2\left(1 - \frac{i\alpha}{Re}\right)\left[-\left(n^2 - \frac{1}{2}\right)\frac{i\alpha}{Re} + n^2\right]\right.$$

$$\left. + \frac{\gamma i\alpha^3}{2}\left[-\left(n^2 - \frac{1}{2}\right)\frac{i\alpha}{Re} + 2n^2\right] + \frac{\gamma^2\alpha^4 n^2}{4}\right\} \qquad (D\text{-}8)$$

```
//WALLACE JOB (2027,0084),'THESIS N = 0',CLASS=B                    PIP00010
//*MAIN LINES=(10)                                                  PIP00020
//  EXEC FRTXCLGP,IMSL=DP,REGION.GO=1024K                           PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*******************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C                                                                   PIP00090
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00100
C                                                                   PIP00110
C       NUMBER N = 0. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00120
C                                                                   PIP00130
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00140
C                                                                   PIP00150
C       COEFFICIENTS FOR THE DERIVATIVES OF Q(R) ONLY, AT THE       PIP00160
C                                                                   PIP00170
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00180
C                                                                   PIP00190
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00200
C                                                                   PIP00210
C       COEFFICIENTS FOR THE DERIVATIVES OF Q(R) AT POINTS          PIP00220
C                                                                   PIP00230
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00240
C                                                                   PIP00250
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00260
C                                                                   PIP00270
C       Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL  RADIAL MESH   PIP00280
C                                                                   PIP00290
C       POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B     PIP00300
C                                                                   PIP00310
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00320
C                                                                   PIP00330
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00340
C                                                                   PIP00350
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00360
C                                                                   PIP00370
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00380
C                                                                   PIP00390
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00400
C                                                                   PIP00410
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00420
C                                                                   PIP00430
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00440
C                                                                   PIP00450
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00460
C                                                                   PIP00470
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00480
```

```
C
C       AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST          PIP00490
C       PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.        PIP00500
C       REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM     PIP00510
C       LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED      PIP00520
C       MESH UP TO A MAXIMUM MESH SIZE OF 50.  THIS PROGRAM MAY BE      PIP00530
C       MODIFIED TO RUN IN THE VM/CMS (TIME SHARING) MODE AS WELL       PIP00540
C       AS THE BATCH MODE (MVS).  WITH APPROPRIATE MODIFICATIONS        PIP00550
C       TO THE PROGRAM DIMENSION STATEMENTS, A MAXIMUM MESH SIZE        PIP00560
C       OF 100 MAY BE USED.  AVERAGE CPU TIME REQUIRED FOR ONE SET      PIP00570
C       OF DATA,  I.E. ONE AXIAL WAVE NUMBER AND ONE REYNOLDS NUMBER    PIP00580
C       IS 8.1 SECONDS.                                                 PIP00590
C*************************************************************        PIP00600
C                                                                       PIP00610
C       DEFINE PROGRAM VARIABLES AND ARRAYS                             PIP00620
C                                                                       PIP00630
      COMPLEX*16 ZERO,ONE,AAXIS(4,3,6),AWALL(4,3,6),A1DER(7),A2DER(7)   PIP00640
      COMPLEX*16 A3DER(7),A4DER(7),A5DER,A(50,50),B(50,50)              PIP00650
      COMPLEX*16 D4(50),D3(50),D2(50),D1(50),D0(50),E2(50),E1(50),E0(50)PIP00660
      COMPLEX*16 D3TEMP,E1TEMP,RAD(52)                                  PIP00670
      COMPLEX*16 EIGA(50),EIGB(50),EIGVEC(50,50),WK(1,1),EIGVAL(50)     PIP00680
      COMPLEX*16 TWO,AA(50,50),BB(50,50),Q(50),DQ(50),U(52)             PIP00690
      COMPLEX*16 QQ6,A6CPX,UNORM(52),ERROR(100),SUMA,SUMB               PIP00700
      REAL*8 UAMP(52),AMPMAX,AMTEMP,UAMNOR(52),A6INV(6)                 PIP00710
      REAL*8 URDP(52),UIDP(52)                                          PIP00720
      REAL*8 GAMR(50),GAMMAX,DMESH                                      PIP00730
      REAL*8 H4TH,H3RD,H2ND,H1ST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA   PIP00740
      REAL*8 SIGN,R2,R3,RE,ALPHA                                        PIP00750
      REAL*8 DTEMPR,DTEMPI,ETEMP,AL22,AL2,AL62,AL6,AL23,ALSQ,AL3RD,AL4THPIP00760
      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)                          PIP00770
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH                     PIP00780
      INTEGER MESH,M,MM,M2,IMAX                                         PIP00790
C                                                                       PIP00800
C       INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS         PIP00810
C                                                                       PIP00820
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,TWO/(2.0D0,0.0D0)/    PIP00830
      DATA AAXIS/72*(0.0D0,0.0D0)/,AWALL/72*(0.0D0,0.0D0)/              PIP00840
```

```
      DATA A1DER/7*(0.0D0,0.0D0)/,A2DER/7*(0.0D0,0.0D0)/                PIP00970
      DATA A3DER/7*(0.0D0,0.0D0)/,A4DER/7*(0.0D0,0.0D0)/                PIP00980
      DATA X1(1)/0.0/,Y1(1)/0.0/,X1(2)/1.0/,Y1(2)/0.0/                  PIP00990
      DATA MESH/50/                                                     PIP01000
C                                                                       PIP01010
CC*****************************************************************   PIP01020
C                                                                       PIP01030
C                              PART I                                   PIP01040
C                                                                       PIP01050
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THE                PIP01060
C                                                                       PIP01070
C        DERIVATIVES OF Q(R) AT INTERIOR MESH POINTS.                   PIP01080
C                                                                       PIP01090
C                                                                       PIP01100
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER       PIP01110
C                                                                       PIP01120
C        DERIVATIVE OF Q(R) AT POINTS  4 - 47                           PIP01130
C                                                                       PIP01140
      DMESH = DFLOAT(MESH)                                              PIP01150
      H4TH = DMESH ** 4                                                 PIP01160
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01170
      A4DER(1) = DCMPLX(ATEMP,0.0D0)                                    PIP01180
      A4DER(7) = A4DER(1)                                               PIP01190
      ATEMP = 2.0D0 * H4TH                                              PIP01200
      A4DER(2) = DCMPLX(ATEMP,0.0D0)                                    PIP01210
      A4DER(6) = A4DER(2)                                               PIP01220
      ATEMP = -6.5D0 * H4TH                                             PIP01230
      A4DER(3) = DCMPLX(ATEMP,0.0D0)                                    PIP01240
      A4DER(5) = A4DER(3)                                               PIP01250
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01260
      A4DER(4) = DCMPLX(ATEMP,0.0D0)                                    PIP01270
C                                                                       PIP01280
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER        PIP01290
C                                                                       PIP01300
C        DERIVATIVE AT POINTS  4 - 47                                   PIP01310
C                                                                       PIP01320
      H3RD = DMESH ** 3                                                 PIP01330
      ATEMP = 0.125D0 * H3RD                                            PIP01340
      A3DER(1) = DCMPLX(ATEMP,0.0D0)                                    PIP01350
      A3DER(7) = -A3DER(1)                                              PIP01360
      A3DER(6) = DCMPLX(H3RD,0.0D0)                                     PIP01370
      A3DER(2) = -A3DER(6)                                              PIP01380
      ATEMP = (13.0D0/8.0D0) * H3RD                                     PIP01390
      A3DER(3) = DCMPLX(ATEMP,0.0D0)                                    PIP01400
      A3DER(5) = -A3DER(3)                                              PIP01410
C                                                                       PIP01420
C        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER       PIP01430
C                                                                       PIP01440
```

```
C         DERIVATIVE AT POINTS  2 - 48                                      PIP01450
C                                                                           PIP01460
      H2ND = DMESH ** 2                                                     PIP01470
      ATEMP = (-1.0D0/12.0D0) * H2ND                                        PIP01480
      A2DER(2) = DCMPLX(ATEMP,0.0D0)                                        PIP01490
      A2DER(6) = A2DER(2)                                                   PIP01500
      ATEMP = (16.0D0/12.0D0) * H2ND                                        PIP01510
      A2DER(3) = DCMPLX(ATEMP,0.0D0)                                        PIP01520
      A2DER(5) = A2DER(3)                                                   PIP01530
      ATEMP = -2.5D0 * H2ND                                                 PIP01540
      A2DER(4) = DCMPLX(ATEMP,0.0D0)                                        PIP01550
C                                                                           PIP01560
CCCC      COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER           PIP01570
CCCC      DERIVATIVE AT POINTS  2 - 48                                      PIP01580
C                                                                           PIP01590
      H1ST = DMESH                                                          PIP01600
      ATEMP = (1.0D0/12.0D0) * H1ST                                         PIP01610
      A1DER(2) = DCMPLX(ATEMP,0.0D0)                                        PIP01620
      A1DER(6) = -A1DER(2)                                                  PIP01630
      ATEMP = (-8.0D0/12.0D0) * H1ST                                        PIP01640
      A1DER(3) = DCMPLX(ATEMP,0.0D0)                                        PIP01650
      A1DER(5) = -A1DER(3)                                                  PIP01660
C                                                                           PIP01670
CC*************************************************************************PIP01680
CCCC                                                                        PIP01690
CCCC                          PART II                                       PIP01700
CCCC                                                                        PIP01710
CCCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR      PIP01720
CCCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.                  PIP01730
CCCC      BOUNDARY CONDITIONS AT THE AXIS ARE: D3Q(0)=0 AND DQ(0)=0.        PIP01740
CCCC      FOR SECOND EQUATION IN Q(R) ONLY, FOR N = 0.                      PIP01750
C                                                                           PIP01760
   51 READ(5,59) I,J,K                                                      PIP01770
      IF(I.EQ.9) GO TO 54                                                   PIP01780
      HTEMP = DMESH ** I                                                    PIP01790
      DO 52 KKK = 1,K                                                       PIP01800
      READ(5,*) AATEMP                                                      PIP01810
      AATEMP = AATEMP * HTEMP                                               PIP01820
      AAXIS(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                                 PIP01830
   52 CONTINUE                                                              PIP01840
      GO TO 51                                                              PIP01850
   54 CONTINUE                                                              PIP01860
C                                                                           PIP01870
CC        READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR      PIP01880
```

```
C
CC          ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.                    PIP01930
CC                                                                             PIP01940
CC          BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1)=0 AND Q(1)=0.           PIP01950
CC                                                                             PIP01960
CC          FOR SECOND EQUATION IN Q(R) ONLY, FOR N = 0.                       PIP01970
C                                                                              PIP01980
   55 READ(5,59) I,J,K                                                         PIP01990
      IF(I.EQ.9) GO TO 57                                                      PIP02000
      HTEMP = DMESH ** I                                                       PIP02010
      JJ = 4 - J                                                               PIP02020
      SIGN = 1.0D0                                                             PIP02030
      IF(I.EQ.1) SIGN = -SIGN                                                  PIP02040
      IF(I.EQ.3) SIGN = -SIGN                                                  PIP02050
      DO 56 KKK = 1,K                                                          PIP02060
      KK = 7 - KKK                                                             PIP02070
      READ(5,*) WATEMP                                                         PIP02080
      WATEMP = WATEMP * HTEMP * SIGN                                           PIP02090
      AWALL(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                                    PIP02100
   56 CONTINUE                                                                 PIP02110
      GO TO 55                                                                 PIP02120
   57 CONTINUE                                                                 PIP02130
   59 FORMAT(I1,1X,I1,1X,I1)                                                   PIP02140
C                                                                              PIP02150
CC          REARRANGING SOME OF THE DATA IN THE AXIS AND WALL                  PIP02160
CC                                                                             PIP02170
CC          COEFFICIENT MATRICES                                               PIP02180
C                                                                              PIP02190
      DO 58 I = 1,6                                                            PIP02200
      AAXIS(2,3,I) = A2DER(I + 1)                                              PIP02210
      AAXIS(1,3,I) = A1DER(I + 1)                                              PIP02220
      AWALL(2,1,I) = A2DER(I)                                                  PIP02230
      AWALL(1,1,I) = A1DER(I)                                                  PIP02240
   58 CONTINUE                                                                 PIP02250
C                                                                              PIP02260
CC          READ IN COEFFICIENTS TO DETERMINE Q(0)                             PIP02270
C                                                                              PIP02280
      DO 10 I = 1,6                                                            PIP02290
      READ (5,*) A6INV(I)                                                      PIP02300
   10 CONTINUE                                                                 PIP02310
C                                                                              PIP02320
C*************************************************************************** PIP02330
C                                                                              PIP02340
CC          READ IN ALPHA AND REYNOLDS NR                                      PIP02350
C                                                                              PIP02360
      IPLOT = 0                                                                PIP02370
   19 CONTINUE                                                                 PIP02380
      READ(5,*) ALPHA,RE                                                       PIP02390
                                                                               PIP02400
```

```
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP02410
C                                                                       PIP02420
C*****************************************************************    PIP02430
C                                                                       PIP02440
C                              PART III                                 PIP02450
C                                                                       PIP02460
C        COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS   PIP02470
C                                                                       PIP02480
C        FOR Q(R) ONLY AND THE RESPECTIVE DERIVATIVES.                  PIP02490
C                                                                       PIP02500
C                                                                       PIP02510
C        PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE COEFFICIENTS      PIP02520
C                                                                       PIP02530
      DTEMPR = (-6.0D0/RE)                                              PIP02540
      D3TEMP = DCMPLX(DTEMPR,0.0D0)                                     PIP02550
      E1TEMP = DCMPLX(-3.0D0,0.0D0)                                     PIP02560
      ALSQ = ALPHA * ALPHA                                              PIP02570
      AL2 = ALPHA + ALPHA                                               PIP02580
      AL22 = ALSQ + ALSQ                                                PIP02590
      AL62 = 6.0D0 * ALSQ                                               PIP02600
      AL6 = 6.0D0 * ALPHA                                               PIP02610
      AL3RD = ALSQ * ALPHA                                              PIP02620
      AL4TH = AL3RD * ALPHA                                             PIP02630
      AL23 = AL3RD + AL3RD                                              PIP02640
C                                                                       PIP02650
      DO 60 I = 1,MESH                                                  PIP02660
      R = (DFLOAT(I) - 0.5D0) / DMESH                                   PIP02670
      R2 = R * R                                                        PIP02680
      R3 = R2 * R                                                       PIP02690
      RAD(I) = DCMPLX(R,0.0D0)                                          PIP02700
      DTEMPR = (-R/RE)                                                  PIP02710
      D4(I) = DCMPLX(DTEMPR,0.0D0)                                      PIP02720
      D3(I) = D3TEMP                                                    PIP02730
      DTEMPR = ((-3.0D0/R) + (AL22 * R))/RE                             PIP02740
      DTEMPI = AL2 * (R - R3)                                           PIP02750
      D2(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02760
      DTEMPR = ((3.0D0/(R2)) + AL62)/RE                                 PIP02770
      DTEMPI = AL6 * (1.0D0 - R2)                                       PIP02780
      D1(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02790
      DTEMPR =-(AL4TH * R)/ RE                                          PIP02800
      DTEMPI = AL23 * (R3 - R)                                          PIP02810
      D0(I) = DCMPLX(DTEMPR,DTEMPI)                                     PIP02820
      ETEMP = -R                                                        PIP02830
      E2(I) = DCMPLX(ETEMP,0.0D0)                                       PIP02840
      E1(I) = E1TEMP                                                    PIP02850
      ETEMP = ALSQ * R                                                  PIP02860
      E0(I) = DCMPLX(ETEMP,0.0D0)                                       PIP02870
   60 CONTINUE                                                          PIP02880
```

```
C                                                                       PIP02890
C*************************************************************************PIP02900
C                                                                       PIP02910
C                                PART IV                                PIP02920
C                                                                       PIP02930
C        COMPUTATION OF THE A AND B MATRIX ELEMENTS.                    PIP02940
C                                                                       PIP02950
C        INITIALIZE THE A AND B MATRICES                                PIP02960
C                                                                       PIP02970
      DO 65 I = 1,MESH                                                  PIP02980
      DO 65 J = 1,MESH                                                  PIP02990
      A(I,J) = ZERO                                                     PIP03000
      B(I,J) = ZERO                                                     PIP03010
   65 CONTINUE                                                          PIP03020
C                                                                       PIP03030
CCC      COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 1,2,3 (AXIS)  PIP03040
C                                                                       PIP03050
      DO 70 J = 1,3                                                     PIP03060
      DO 70 K = 1,6                                                     PIP03070
      AODER = ZERO                                                      PIP03080
      IF(K.EQ.J) AODER = ONE                                            PIP03090
      A(J,K) = (AAXIS(4,J,K)*D4(J) + AAXIS(3,J,K)*D3(J) +               PIP03100
     1          AAXIS(2,J,K)*D2(J) + AAXIS(1,J,K)*D1(J) + AODER*DO(J))  PIP03110
      B(J,K) = (AAXIS(2,J,K)*E2(J) + AAXIS(1,J,K)*E1(J) + AODER*EO(J))  PIP03120
   70 CONTINUE                                                          PIP03130
C                                                                       PIP03140
CCC      COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 4 - 47        PIP03150
C                                                                       PIP03160
      M = MESH - 3                                                      PIP03170
      DO 80 J = 4,M                                                     PIP03180
      DO 80 K = 1,7                                                     PIP03190
      L = J - 4 + K                                                     PIP03200
      AODER = ZERO                                                      PIP03210
      IF(L.EQ.J) AODER = ONE                                            PIP03220
      A(J,L) = (A4DER(K)*D4(J) + A3DER(K)*D3(J) +                       PIP03230
     1          A2DER(K)*D2(J) + A1DER(K)*D1(J) + AODER*DO(J))          PIP03240
      B(J,L) = (A2DER(K)*E2(J) + A1DER(K)*E1(J) + AODER*EO(J))          PIP03250
   80 CONTINUE                                                          PIP03260
C                                                                       PIP03270
CCC      COMPUTATION OF A AND B MATRIX ELEMENTS AT POINTS 48 - 50 (WALL)PIP03280
C                                                                       PIP03290
      MM = M - 3                                                        PIP03300
      DO 90 J = 1,3                                                     PIP03310
      L = J + M                                                         PIP03320
      DO 90 K = 1,6                                                     PIP03330
      N = MM + K                                                        PIP03340
      AODER = ZERO                                                      PIP03350
      IF(L.EQ.N) AODER = ONE                                            PIP03360
```

```
      A(L,N) = (AWALL(4,J,K)*D4(L) + AWALL(3,J,K)*D3(L) +                    PIP03370
     1          AWALL(2,J,K)*D2(L) + AWALL(1,J,K)*D1(L) + AODER*DO(L))       PIP03380
      B(L,N) = (AWALL(2,J,K)*E2(L) + AWALL(1,J,K)*E1(L) + AODER*EO(L))       PIP03390
   90 CONTINUE                                                               PIP03400
C                                                                            PIP03410
C*************************************************************************** PIP03420
C                                                                            PIP03430
C                                  PART V                                    PIP03440
C                                                                            PIP03450
C        COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS              PIP03460
C                                                                            PIP03470
C        USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT             PIP03480
C                                                                            PIP03490
C        EQUATIONS IN THE FORM  A * K = GAMMA * B * X.                       PIP03500
C                                                                            PIP03510
C                                                                            PIP03520
C             INITIALIZE EIGENVECTOR AND EIGENVALUE ARRAYS.                  PIP03530
C                                                                            PIP03540
      DO 140 I = 1,50                                                        PIP03550
      EIGVAL(I) = ZERO                                                       PIP03560
      EIGA(I) = ZERO                                                         PIP03570
      EIGB(I) = ZERO                                                         PIP03580
      DO 140 J = 1,50                                                        PIP03590
      EIGVEC(I,J) = ZERO                                                     PIP03600
      AA(I,J) = A(I,J)                                                       PIP03610
      BB(I,J) = B(I,J)                                                       PIP03620
  140 CONTINUE                                                               PIP03630
      WRITE(6,145)                                                           PIP03640
  145 FORMAT(1X,//,1X,'*****************************',                       PIP03650
     1   '*****************************************',                       PIP03660
     2   '*****************************************',//)                    PIP03670
      IJOB = 1                                                               PIP03680
      CALL EIGZC (AA,50,BB,50,MESH,IJOB,EIGA,EIGB,EIGVEC,50,WK,INEER,IER)    PIP03690
      IF(INEER.EQ.0) WRITE(6,150)                                            PIP03700
  150 FORMAT(1X,/////,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',            PIP03710
     1       'ITERATIONS',//)                                                PIP03720
      IF(INEER.NE.0) WRITE(6,155) INEER                                      PIP03730
  155 FORMAT(1X,/////,10X,'FAILED TO CONVERGE ON THE',I4,1X,                 PIP03740
     1       'TH EIGENVALUE',//)                                             PIP03750
      DO 165 I = 1,MESH                                                      PIP03760
      EIGVAL(I) = EIGA(I)/EIGB(I)                                            PIP03770
      GAMR(I) = DREAL(EIGVAL(I))                                             PIP03780
  165 CONTINUE                                                               PIP03790
      NGAM = 1                                                               PIP03800
      GAMMAX = GAMR(1)                                                       PIP03810
      DO 170 I = 2,MESH                                                      PIP03820
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                        PIP03830
      GAMMAX = GAMR(I)                                                       PIP03840
```

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      NGAM = I                                                          PIP03850
  169 CONTINUE                                                          PIP03860
  170 CONTINUE                                                          PIP03870
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                            PIP03880
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,  PIP03890
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)      PIP03900
C                                                                       PIP03910
CC*********************************************************************PIP03920
C                                                                       PIP03930
C                              PART VI                                  PIP03940
C                                                                       PIP03950
C        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING      PIP03960
C                                                                       PIP03970
C        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL         PIP03980
C                                                                       PIP03990
C        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.               PIP04000
C                                                                       PIP04010
      DO 351 I = 1,MESH                                                 PIP04020
      SUMA = ZERO                                                       PIP04030
      SUMB = ZERO                                                       PIP04040
      DO 350 J = 1,MESH                                                 PIP04050
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                         PIP04060
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                         PIP04070
  350 CONTINUE                                                          PIP04080
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                         PIP04090
  351 CONTINUE                                                          PIP04100
      WRITE(6,375)                                                      PIP04110
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)           PIP04120
      WRITE(6,380) (ERROR(I),I = 1,MESH)                                PIP04130
  380 FORMAT(1X,2D30.18)                                                PIP04140
C                                                                       PIP04150
CC*********************************************************************PIP04160
C                                                                       PIP04170
C                              PART VII                                 PIP04180
C                                                                       PIP04190
C        COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).              PIP04200
C                                                                       PIP04210
C                                                                       PIP04220
C        OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE       PIP04230
C                                                                       PIP04240
C        EIGENVALUE,  GAMMA*.                                           PIP04250
C                                                                       PIP04260
      WRITE(6,398)                                                      PIP04270
  398 FORMAT(1X,///,5X,'EIGENVECTOR Q(I) CORRESPONDING TO THE LEAST ',  PIP04280
     1       'STABLE EIGENVALUE',/)                                     PIP04290
      DO 400 I = 1,MESH                                                 PIP04300
      Q(I) = EIGVEC(I,NGAM)                                             PIP04310
      WRITE(6,399) Q(I)                                                 PIP04320
```

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  399 FORMAT(5X,D30.18,3X,D30.18)                                       PIP04330
  400 CONTINUE                                                          PIP04340
C                                                                       PIP04350
C     COMPUTATION OF DQ(R)                                              PIP04360
C                                                                       PIP04370
      DO 410 J = 1,3                                                    PIP04380
      DQ(J) = ZERO                                                      PIP04390
      DO 410 K = 1,6                                                    PIP04400
      DQ(J) = DQ(J) + Q(K) * AAXIS(1,J,K)                               PIP04410
  410 CONTINUE                                                          PIP04420
C                                                                       PIP04430
      M = MESH - 3                                                      PIP04440
      DO 420 J = 4,M                                                    PIP04450
      DQ(J) = ZERO                                                      PIP04460
      DO 420 K = 1,7                                                    PIP04470
      L = J - 4 + K                                                     PIP04480
      DQ(J) = DQ(J) + Q(L) * A1DER(K)                                   PIP04490
  420 CONTINUE                                                          PIP04500
C                                                                       PIP04510
      MM = M - 3                                                        PIP04520
      DO 430 J = 1,3                                                    PIP04530
      L = J + M                                                         PIP04540
      DQ(L) = ZERO                                                      PIP04550
      DO 430 K = 1,6                                                    PIP04560
      N = MM + K                                                        PIP04570
      DQ(L) = DQ(L) + Q(N) * AWALL(1,J,K)                               PIP04580
  430 CONTINUE                                                          PIP04590
C                                                                       PIP04600
      DO 440 I = 1,MESH                                                 PIP04610
      U(I) = Q(I) * TWO + RAD(I) * DQ(I)                                PIP04620
  440 CONTINUE                                                          PIP04630
C                                                                       PIP04640
C        COMPUTE Q(0)                                                   PIP04650
C                                                                       PIP04660
      Q06 = ZERO                                                        PIP04670
      DO 441 I = 1,6                                                    PIP04680
      A6CPX = DCMPLX(A6INV(I),0.0D0)                                    PIP04690
      Q06 = Q06 + ( A6CPX * Q(I) )                                      PIP04700
  441 CONTINUE                                                          PIP04710
C                                                                       PIP04720
C        REARRANGE DATA FOR PLOT ROUTINE                                PIP04730
C                                                                       PIP04740
      M2 = MESH + 2                                                     PIP04750
      DO 450 I = 1,MESH                                                 PIP04760
      II = M2 - I                                                       PIP04770
      III = II - 1                                                      PIP04780
      U(II) = U(III)                                                    PIP04790
      RAD(II) = RAD(III)                                                PIP04800
```

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  450 CONTINUE                                                          PIP04810
      RAD(1) = ZERO                                                     PIP04820
      RAD(M2) = ONE                                                     PIP04830
      U(1) = Q06 * TWO                                                  PIP04840
      U(M2) = ZERO                                                      PIP04850
      DO 460 I = 1,M2                                                   PIP04860
      RADR(I) = DREAL(RAD(I))                                           PIP04870
  460 CONTINUE                                                          PIP04880
C                                                                       PIP04890
CC        COMPUTE THE NORMALIZED PERTURBATION VELOCITY ANDAMPLITUDE     PIP04900
C                                                                       PIP04910
      DO 465 I = 1,M2                                                   PIP04920
      UAMP(I) = CDABS(U(I))                                             PIP04930
  465 CONTINUE                                                          PIP04940
      AMPMAX = UAMP(1)                                                  PIP04950
      IMAX = 1                                                          PIP04960
      DO 470 J = 2,M2                                                   PIP04970
      AMTEMP = UAMP(J)                                                  PIP04980
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                    PIP04990
      AMPMAX = UAMP(J)                                                  PIP05000
      IMAX = J                                                          PIP05010
  471 CONTINUE                                                          PIP05020
  470 CONTINUE                                                          PIP05030
      WRITE(6,472)                                                      PIP05040
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',             PIP05050
     124X,'NORMALIZED AMPLITUDE',/)                                     PIP05060
      DO 475 K = 1,M2                                                   PIP05070
      UAMNOR(K) = UAMP(K) / AMPMAX                                      PIP05080
      AMPNOR(K) = UAMNOR(K)                                             PIP05090
      UNORM(K) = U(K) / U(IMAX)                                         PIP05100
      URDP(K) = DREAL(UNORM(K))                                         PIP05110
      UR(K) = URDP(K)                                                   PIP05120
      UIDP(K) = DIMAG(UNORM(K))                                         PIP05130
      UI(K) = UIDP(K)                                                   PIP05140
      KK = K - 1                                                        PIP05150
      WRITE(6,466) KK,URDP(K),JIDP(K),AMPNOR(K)                         PIP05160
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                PIP05170
  475 CONTINUE                                                          PIP05180
C                                                                       PIP05190
CC*********************************************************************PIP05200
CC                                                                      PIP05210
CC                              PART VIII                               PIP05220
CC                                                                      PIP05230
CC        PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.     PIP05240
C                                                                       PIP05250
      IPLOT = IPLOT + 1                                                 PIP05260
      RMESH = FLOAT(MESH)                                               PIP05270
      ALPLOT = ALPHA                                                    PIP05280
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      REPLOT = RE                                                          PIP05290
      GAMPLT = GAMMAX                                                      PIP05300
      CALL PLOTG(RADR,UR,M2,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)      PIP05310
      CALL PLOTG(RADR,UI,M2,2,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)      PIP05320
C     CALL PLOTG(RADR,AMPNDR,M2,3,1,' ',1,' ',1,0.1,-1,1.5,5.)             PIP05330
      CALL PLOTG(X1,Y1,2,4,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)       PIP05340
      XO = 0.15                                                            PIP05350
      YO = 5.6                                                             PIP05360
      IF(IPLOT.EQ.4) IPLOT = 1                                             PIP05370
      IF(IPLOT.EQ.2) YO = 12.1                                             PIP05380
      IF(IPLOT.EQ.3) YO = 18.6                                             PIP05390
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                        PIP05400
      GO TO 19                                                             PIP05410
  180 CONTINUE                                                             PIP05420
      CALL PLOT(0.0,0.0,999)                                               PIP05430
      STOP                                                                 PIP05440
      END                                                                  PIP05450
C                                                                          PIP05460
CC    ..................................................................   PIP05470
CCC                                                                        PIP05480
CCC   SUBROUTINE TITLE1                                                    PIP05490
CCC                                                                        PIP05500
CCC      SUBROUTINE TO WRITE TITLE ON PLOTS                                PIP05510
C                                                                          PIP05520
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                  PIP05530
      RN = 0.0                                                             PIP05540
      YHT = 0.14                                                           PIP05550
      CALL NEWPEN(2)                                                       PIP05560
      CALL SYMBOL(XO,YO,YHT,                                               PIP05570
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)                 PIP05580
      YHT = 0.08                                                           PIP05590
      DELY = 0.20                                                          PIP05600
      CALL NEWPEN(1)                                                       PIP05610
      YO = YO - 0.8                                                        PIP05620
      XO = XO + 2.5                                                        PIP05630
      XN = XO + .75                                                        PIP05640
      CALL SYMBOL(XO,YO,YHT,'N      = ',0.0,9)                             PIP05650
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                     PIP05660
      YO = YO - DELY                                                       PIP05670
      CALL SYMBOL(XO,YO,YHT,'MESH   = ',0.0,9)                             PIP05680
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                                  PIP05690
      YO = YO - DELY                                                       PIP05700
      CALL SYMBOL(XO,YO,YHT,'ALPHA  = ',0.0,9)                             PIP05710
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                                  PIP05720
      YO = YO - DELY                                                       PIP05730
      CALL SYMBOL(XO,YO,YHT,'REY NR = ',0.0,9)                             PIP05740
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                                  PIP05750
      YO = YO - DELY                                                       PIP05760
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      CALL SYMBOL(XO,YO,YHT,'GAMMA* = ',0.0,9)                                  PIP05770
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                                       PIP05780
      YO = YO - 3.4                                                             PIP05790
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                         PIP05800
      YO = YO - DELY                                                            PIP05810
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                         PIP05820
      YO = YO - DELY                                                            PIP05830
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                         PIP05840
C                                                                               PIP05850
C        WRITE THE AXIS LABLES                                                  PIP05860
C                                                                               PIP05870
      XO = XO - 2.55                                                            PIP05880
      YO = YO + 0.3                                                             PIP05890
      YHT = 0.12                                                                PIP05900
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)          PIP05910
      XO = XO + 2.8                                                             PIP05920
      YO = YO - 1.0                                                             PIP05930
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                               PIP05940
      RETURN                                                                    PIP05950
      END                                                                       PIP05960
/*                                                                              PIP05970
//GO.SYSIN DD *                                                                 PIP05980
4 1 6                                                                           PIP05990
      0.14905527754213541 1D+01                                                 PIP06000
     -0.14672855130067858 9D+01                                                 PIP06010
     -0.95280803446403548 3D+00                                                 PIP06020
      0.13147481450175468 9D+01                                                 PIP06030
     -0.44306626816295204 0D+00                                                 PIP06040
      0.57858895194873314 1D-01                                                 PIP06050
4 2 6                                                                           PIP06060
     -0.52997087407008978 1D+01                                                 PIP06070
      0.10923281172878383 0D+02                                                 PIP06080
     -0.80354336201689449 1D+01                                                 PIP06090
      0.27666474282402671 0D+01                                                 PIP06100
     -0.38098574438380006 2D+00                                                 PIP06110
      0.26199504134987103 3D-01                                                 PIP06120
4 3 6                                                                           PIP06130
      0.18299535726319717 3D+01                                                 PIP06140
     -0.64925902480678203 7D+01                                                 PIP06150
      0.93268795274362297 2D+01                                                 PIP06160
     -0.64967929150805332 5D+01                                                 PIP06170
      0.19991083579918473 9D+01                                                 PIP06180
     -0.16655829491175344 5D+00                                                 PIP06190
3 1 5                                                                           PIP06200
      0.30710207612456812 4D+01                                                 PIP06210
     -0.55170415224913651 5D+01                                                 PIP06220
      0.31907439446366840 9D+01                                                 PIP06230
     -0.84801038062284062 0D+00                                                 PIP06240
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  0.10328719723183425 7D+00                                                PIP06250
3 2 5                                                                      PIP06260
  0.90960207612456422 9D+00                                                PIP06270
 -0.44420415224913223 4D+00                                                PIP06280
 -0.12184256055363442 7D+01                                                PIP06290
  0.85769896193771688 6D+00                                                PIP06300
 -0.10467128027681674 2D+00                                                PIP06310
3 3 5                                                                      PIP06320
 -0.29096020761245817 7D+01                                                PIP06330
  0.64442041522491599 9D+01                                                PIP06340
 -0.47815743944636999 2D+01                                                PIP06350
  0.11423010380622855 5D+01                                                PIP06360
  0.10467128027681571 5D+00                                                PIP06370
2 1 4                                                                      PIP06380
 -0.12228993261989693 5D+01                                                PIP06390
  0.13512187871581455 8D+01                                                PIP06400
 -0.13956599286563649 3D+00                                                PIP06410
  0.11246531906460608 5D-01                                                PIP06420
2 2 4                                                                      PIP06430
  0.12497522790328994 3D+01                                                PIP06440
 -0.24995541022592189 2D+01                                                PIP06450
  0.13330856123662322 4D+01                                                PIP06460
 -0.83283789139913064 2D-01                                                PIP06470
1 1 3                                                                      PIP06480
 -0.66666666666666668 5D+00                                                PIP06490
  0.75000000000000000 0D+00                                                PIP06500
 -0.83333333333333314 8D-01                                                PIP06510
1 2 3                                                                      PIP06520
 -0.99999999999999977 8D+00                                                PIP06530
  0.74999999999999977 3D+00                                                PIP06540
  0.25000000000000013 9D+00                                                PIP06550
9 0 0                                                                      PIP06560
4 1 6                                                                      PIP06570
  0.11500000000000071 4D+03                                                PIP06580
 -0.38444444444445025 5D+02                                                PIP06590
  0.16400000000000385 8D+02                                                PIP06600
 -0.52244897959184979 4D+01                                                PIP06610
  0.11234567901234950 9D+01                                                PIP06620
 -0.11570247933884755 8D+00                                                PIP06630
4 2 6                                                                      PIP06640
 -0.26000000000000802 9D+02                                                PIP06650
  0.17666666666667016 0D+02                                                PIP06660
 -0.11200000000000230 2D+02                                                PIP06670
  0.39591836734695107 2D+01                                                PIP06680
 -0.66666666666670049 1D+00                                                PIP06690
  0.57851239669425180 7D-01                                                PIP06700
4 3 6                                                                      PIP06710
  0.10000000000008526 5D+01                                                PIP06720
```

```
      -0.62222222222280408D+01                                          PIP06730
       0.92000000000045759D+01                                          PIP06740
      -0.64489795918368990 5D+01                                        PIP06750
       0.19876543209876977 9D+01                                        PIP06760
      -0.16528925619835255 2D+00                                        PIP06770
3 1 5                                                                   PIP06780
      -0.17499999999999975 1D+02                                        PIP06790
       0.11111111111110760 3D+00                                        PIP06800
       0.12000000000000137 2D+01                                        PIP06810
      -0.34693877551020739 5D+00                                        PIP06820
       0.43209876543210391 1D-01                                        PIP06830
3 2 5                                                                   PIP06840
       0.13499999999999968 0D+02                                        PIP06850
      -0.38888888888888821 3D+01                                        PIP06860
      -0.14210854715202003 7D-13                                        PIP06870
       0.55102040816326658 4D+00                                        PIP06880
      -0.67901234567901785 9D-01                                        PIP06890
3 3 5                                                                   PIP06900
      -0.15500000000000113 7D+02                                        PIP06910
       0.98888888888889106 0D+01                                        PIP06920
      -0.60000000000000532 9D+01                                        PIP06930
       0.14489795918367498 5D+01                                        PIP06940
       0.67901234567899093 6D-01                                        PIP06950
2 1 4                                                                   PIP06960
      -0.46666666666666585 0D+01                                        PIP06970
       0.21111111111111102 7D+01                                        PIP06980
      -0.31999999999999993 3D+00                                        PIP06990
       0.34013605442176852 1D-01                                        PIP07000
2 2 4                                                                   PIP07010
       0.99999999999997868 4D+00                                        PIP07020
      -0.24444444444444493 0D+01                                        PIP07030
       0.13200000000000011 7D+01                                        PIP07040
      -0.81632653061224511 0D-01                                        PIP07050
1 1 3                                                                   PIP07060
       0.24999999999999988 9D+01                                        PIP07070
       0.22222222222222229 3D+00                                        PIP07080
      -0.19999999999999978 0D-01                                        PIP07090
1 2 3                                                                   PIP07100
      -0.45000000000000035 5D+01                                        PIP07110
       0.13333333333333330 4D+01                                        PIP07120
       0.18000000000000007 0D+00                                        PIP07130
9 0 0                                                                   PIP07140
       0.12791408217393292 9D+01                                        PIP07150
      -0.41669653599285105 8D+00                                        PIP07160
       0.19554923111095109 9D+00                                        PIP07170
      -0.73690837175280396 5D-01                                        PIP07180
       0.17653187627415903 9D-01                                        PIP07190
      -0.19558675041653811 8D-02                                        PIP07200
```

```
0.0D0     575.0D0                                                   PIP07210
0.5D0     575.0D0                                                   PIP07220
1.0D0     575.0D0                                                   PIP07230
2.0D0     575.0D0                                                   PIP07240
4.0D0     575.0D0                                                   PIP07250
8.0D0     575.0D0                                                   PIP07260
16.0D0    575.0D0                                                   PIP07270
32.0D0    575.0D0                                                   PIP07280
0.0D0     1150.0D0                                                  PIP07290
0.5D0     1150.0D0                                                  PIP07300
1.0D0     1150.0D0                                                  PIP07310
2.0D0     1150.0D0                                                  PIP07320
4.0D0     1150.0D0                                                  PIP07330
8.0D0     1150.0D0                                                  PIP07340
16.0D0    1150.0D0                                                  PIP07350
32.0D0    1150.0D0                                                  PIP07360
-1.0D0 1.0D0                                                        PIP07370
/*                                                                  PIP07380
//                                                                  PIP07390
```

```
//WALLACE JOB (2027,0084),'THESIS N = 1',CLASS=C                    PIP00010
//*MAIN LINES=(10)                                                  PIP00020
//  EXEC FRTXCLGP,IMSL=DP,REGION.GO=2048K                           PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*****************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00090
C       NUMBER N = 1. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00100
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00110
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT THE    PIP00120
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00130
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00140
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT POINTS PIP00150
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00160
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00170
C       P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL RADIAL PIP00180
C       MESH POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B PIP00190
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00200
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00210
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00220
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00230
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00240
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00250
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00260
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00270
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00280
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C          AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST          PIP00490
C          PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.        PIP00500
C          REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM     PIP00510
C          LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED      PIP00520
C          MESH UP TO A MAXIMUM MESH SIZE OF 48.  BECAUSE OF THE           PIP00530
C          LARGE AMOUNT OF MEMORY REQUIRED TO RUN THIS PROGRAM,  IT        PIP00540
C          CAN ONLY BE RUN IN THE BATCH MODE (MVS).  AVERAGE CPU TIME      PIP00550
C          REQUIRED FOR ONE SET OF DATA,  I.E. ONE AXIAL WAVE NUMBER       PIP00560
C          AND ONE REYNOLDS NUMBER,  IS 58 SECONDS.                        PIP00570
C*****************************************************************       PIP00580
C                                                                          PIP00590
C          DEFINE PROGRAM VARIABLES AND ARRAYS.                            PIP00600
C                                                                          PIP00610
      COMPLEX*16 A(100,100),B(100,100)                                     PIP00620
      COMPLEX*16 ZERO,ONE,AAXISQ(4,3,6),AWALLQ(4,3,6),A1DERQ(7)            PIP00630
      COMPLEX*16 A2DERQ(7),A3DERQ(7),A4DERQ(7),A0DER                       PIP00640
      COMPLEX*16 D3Q1(50),D2Q1(50),D1Q1(50),D0Q1(50)                       PIP00650
      COMPLEX*16 D4Q2(50),D3Q2(50),D2Q2(50),D1Q2(50),D0Q2(50)              PIP00660
      COMPLEX*16 E1Q1(50),E0Q1(50),E2Q2(50),E1Q2(50),E0Q2(50)              PIP00670
      COMPLEX*16 AAXISP(2,2,5),AWALLP(2,2,5),A1DERP(5),A2DERP(5)           PIP00680
      COMPLEX*16 D2P1(50),D1P1(50),D0P1(50),D2P2(50),D1P2(50)              PIP00690
      COMPLEX*16 D0P2(50),E0P1(50),E0P2(50)                                PIP00700
      COMPLEX*16 EIGA(100),EIGB(100),EIGVEC(100,100),EIGVAL(100)           PIP00710
      COMPLEX*16 AA(100,100),BB(100,100),ONEIMG,TWO,THREE                  PIP00720
      COMPLEX*16 Q(50),DQ(50),P(50),J(52),RAD(52),UNORM(52)                PIP00730
      COMPLEX*16 AI5P(6),AI5D2P(6),AI6D2Q(6),H0A1(52),H0A2(52)             PIP00740
      COMPLEX*16 H0B1(52),H0B2(52),H0,Q0                                   PIP00750
      COMPLEX*16 H0D1PA,H0D2PA,Q0D1QA,Q0D2QA,Q0D3QA,Q0D1QB,BETA            PIP00760
      COMPLEX*16 Q0D4QA,Q0D2QB,Q0TEM1,Q0TEM2,H0D1QA,H0D2QA,H0D3QA          PIP00770
      COMPLEX*16 H0D4QA,H0D1QB,H0D2QB,AAXQ1,AAXQ2,AWAQ1,AWAQ2              PIP00780
      COMPLEX*16 D2P01,D2P02,D2Q01,D2Q02,D0P01,D0P02,D0Q01,D0Q02           PIP00790
      COMPLEX*16 D0H01,D0H02,E0P01,E0P02,E0Q01,E0Q02,E0H01,E0H02           PIP00800
      COMPLEX*16 ERROR(100),WK(1,1),SUMA,SUMB                              PIP00810
      REAL*8 GAMR(100),GAMMAX,DMESH                                        PIP00820
      REAL*8 H4TH,H3RD,H2ND,HIST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA      PIP00830
      REAL*8 TEMPR,TEMPI,SIGN,R2,R3,R4                                     PIP00840
      REAL*8 AL2,AL4,AL10,ALSQ,AL4TH,ALSQ2,ALSQ4,ALSQ5,ALSQ10,AL3RD2       PIP00850
      REAL*8 UAMP(52),AMPHAX,AMTEMP,UAMNOR(52),URDP(52),UIDP(52)           PIP00860
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      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)                          PIP00970
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH                     PIP00980
      INTEGER MESH,M,MM,M2,IMAX,MP,M1P,MESH2,MH0,MQ0,NGAM               PIP00990
C                                                                       PIP01000
CCC      INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS        PIP01010
C                                                                       PIP01020
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,TWO/(2.0D0,0.0D0)/    PIP01030
      DATA ONEIMG/(0.0D0,1.0D0)/                                        PIP01040
      DATA THREE/(3.0D0,0.0D0)/                                         PIP01050
      DATA AAXISQ/72*(0.0D0,0.0D0)/,AWALLQ/72*(0.0D0,0.0D0)/            PIP01060
      DATA A1DERQ/7*(0.0D0,0.0D0)/,A2DERQ/7*(0.0D0,0.0D0)/              PIP01070
      DATA A3DERQ/7*(0.0D0,0.0D0)/,A4DERQ/7*(0.0D0,0.0D0)/              PIP01080
      DATA AAXISP/20*(0.0D0,0.0D0)/,AWALLP/20*(0.0D0,0.0D0)/            PIP01090
      DATA A1DERP/5*(0.0D0,0.0D0)/,A2DERP/5*(0.0D0,0.0D0)/              PIP01100
      DATA X1(1)/0.0/,X1(2)/1.0/,Y1(1)/0.0/,Y1(2)/0.0/                  PIP01110
      DATA MESH/48/                                                     PIP01120
C                                                                       PIP01130
C*************************************************************        PIP01140
C                                                                       PIP01150
C                              PART I                                   PIP01160
C                                                                       PIP01170
CCC      COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER       PIP01180
CCCC                                                                    PIP01190
CCC      DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01200
C                                                                       PIP01210
      DMESH = DFLOAT(MESH)                                              PIP01220
      H4TH = DMESH ** 4                                                 PIP01230
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01240
      A4DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01250
      A4DERQ(7) = A4DERQ(1)                                             PIP01260
      ATEMP = 2.0D0 * H4TH                                              PIP01270
      A4DERQ(2) = DCMPLX(ATEMP,0.0D0)                                   PIP01280
      A4DERQ(6) = A4DERQ(2)                                             PIP01290
      ATEMP = -6.5D0 * H4TH                                             PIP01300
      A4DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01310
      A4DERQ(5) = A4DERQ(3)                                             PIP01320
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01330
      A4DERQ(4) = DCMPLX(ATEMP,0.0D0)                                   PIP01340
C                                                                       PIP01350
CCC      COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER        PIP01360
CCCC                                                                    PIP01370
CCC      DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01380
C                                                                       PIP01390
      H3RD = DMESH ** 3                                                 PIP01400
      ATEMP = 0.125D0 * H3RD                                            PIP01410
      A3DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01420
      A3DERQ(7) = -A3DERQ(1)                                            PIP01430
      A3DERQ(6) = DCMPLX(H3RD,0.0D0)                                    PIP01440
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      A3DERQ(2) = -A3DERQ(5)                                                PIP01450
      ATEMP = (13.0D0/8.0D0) * H3RD                                         PIP01460
      A3DERQ(3) = DCMPLX(ATEMP,0.0D0)                                       PIP01470
      A3DERQ(5) = -A3DERQ(3)                                                PIP01480
C                                                                           PIP01490
CCCCC    COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER           PIP01500
C                                                                           PIP01510
CCCCC    DERIVATIVES OF P(R) AND Q(R) AT INTERIOR MESH POINTS.              PIP01520
C                                                                           PIP01530
      H2ND = DMESH ** 2                                                     PIP01540
      ATEMP = (-1.0D0/12.0D0) * H2ND                                        PIP01550
      A2DERQ(2) = DCMPLX(ATEMP,0.0D0)                                       PIP01560
      A2DERQ(6) = A2DERQ(2)                                                 PIP01570
      ATEMP = (16.0D0/12.0D0) * H2ND                                        PIP01580
      A2DERQ(3) = DCMPLX(ATEMP,0.0D0)                                       PIP01590
      A2DERQ(5) = A2DERQ(3)                                                 PIP01600
      ATEMP = -2.5D0 * H2ND                                                 PIP01610
      A2DERQ(4) = DCMPLX(ATEMP,0.0D0)                                       PIP01620
      A2DERP(1) = A2DERQ(2)                                                 PIP01630
      A2DERP(2) = A2DERQ(3)                                                 PIP01640
      A2DERP(3) = A2DERQ(4)                                                 PIP01650
      A2DERP(4) = A2DERQ(5)                                                 PIP01660
      A2DERP(5) = A2DERQ(5)                                                 PIP01670
C                                                                           PIP01680
CCCCC    COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER            PIP01690
C                                                                           PIP01700
CCCCC    DERIVATIVES OF P(R) AND Q(R) AT INTERIOR MESH POINTS.              PIP01710
C                                                                           PIP01720
      H1ST = DMESH                                                          PIP01730
      ATEMP = (1.0D0/12.0D0) * H1ST                                         PIP01740
      A1DERQ(2) = DCMPLX(ATEMP,0.0D0)                                       PIP01750
      A1DERQ(6) = -A1DERQ(2)                                                PIP01760
      ATEMP = (-8.0D0/12.0D0) * H1ST                                        PIP01770
      A1DERQ(3) = DCMPLX(ATEMP,0.0D0)                                       PIP01780
      A1DERQ(5) = -A1DERQ(3)                                                PIP01790
      A1DERP(1) = A1DERQ(2)                                                 PIP01800
      A1DERP(2) = A1DERQ(3)                                                 PIP01810
      A1DERP(4) = A1DERQ(5)                                                 PIP01820
      A1DERP(5) = A1DERQ(5)                                                 PIP01830
C*****************************************************************        PIP01840
C                                                                           PIP01850
C                                                                           PIP01860
C                          PART II                                          PIP01870
C                                                                           PIP01880
CCCCCCCC READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR       PIP01890
C                                                                           PIP01900
CCCCC    ALL DERIVATIVES OF P(R) AT POINTS NEAR THE AXIS.                   PIP01910
C                                                                           PIP01920
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C         BOUNDARY CONDITIONS AT THE AXIS ARE: DP(0) = 0 AND Q(0) = VAR.  PIP01930
C                                                                         PIP01940
   41 READ(5,49) I,J,K                                                    PIP01950
      IF(I.EQ.9) GO TO 44                                                 PIP01960
      HTEMP = DMESH ** I                                                  PIP01970
      DO 42 KKK = 1,K                                                     PIP01980
      READ(5,*) AATEMP                                                    PIP01990
      AATEMP = AATEMP * HTEMP                                             PIP02000
      AAXISP(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                              PIP02010
   42 CONTINUE                                                            PIP02020
      GO TO 41                                                            PIP02030
   44 CONTINUE                                                            PIP02040
C                                                                         PIP02050
CCCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS        PIP02060
CCCC                                                                      PIP02070
CCC       FOR P(0) AND D2P(J).                                            PIP02080
C                                                                         PIP02090
      DO 70 I = 1,5                                                       PIP02100
      READ(5,*) AATEMP                                                    PIP02110
      AI5P(I) = DCMPLX(AATEMP,0.0D0)                                      PIP02120
   70 CONTINUE                                                            PIP02130
      AI5P(6) = ZERO                                                      PIP02140
      DO 71 I = 1,5                                                       PIP02150
      READ(5,*) AATEMP                                                    PIP02160
      AATEMP = AATEMP * (DMESH ** 2)                                      PIP02170
      AI5D2P(I) = DCMPLX(AATEMP,0.0D0)                                    PIP02180
   71 CONTINUE                                                            PIP02190
      AI5D2P(6) = ZERO                                                    PIP02200
C                                                                         PIP02210
CCCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR    PIP02220
CCCCC                                                                     PIP02230
CCCCC     ALL DERIVATIVES OF P(R) AT POINTS NEAR THE WALL.                PIP02240
CCCCC                                                                     PIP02250
CCCC      BOUNDARY CONDITION AT THE WALL IS:  P(1) = I * H(0)             PIP02260
C                                                                         PIP02270
   45 READ(5,49) I,J,K                                                    PIP02280
      IF(I.EQ.9) GO TO 47                                                 PIP02290
      HTEMP = DMESH ** I                                                  PIP02300
      JJ = 3 - J                                                          PIP02310
      SIGN = 1.0D0                                                        PIP02320
      IF(I.EQ.1) SIGN = -SIGN                                             PIP02330
      DO 46 KKK = 1,K                                                     PIP02340
      KK = 6 - KKK                                                        PIP02350
      READ(5,*) WATEMP                                                    PIP02360
      WATEMP = WATEMP * HTEMP * SIGN                                      PIP02370
      AWALLP(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                              PIP02380
   46 CONTINUE                                                            PIP02390
      GO TO 45                                                            PIP02400
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   47 CONTINUE                                                          PIP02410
   49 FORMAT(I1, 1X.I1,1X,I1)                                           PIP02420
C                                                                       PIP02430
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP02440
CCC                                                                     PIP02450
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.               PIP02460
CCC                                                                     PIP02470
CCC      BOUNDARY CONDITION AT THE AXIS IS: DQ(0) = 0.                  PIP02480
C                                                                       PIP02490
   51 READ(5,49) I,J,K                                                  PIP02500
      IF(I.EQ.9) GO TO 54                                               PIP02510
      HTEMP = DMESH ** I                                                PIP02520
      DO 52 KKK = 1,K                                                   PIP02530
      READ(5,*) AATEMP                                                  PIP02540
      AATEMP = AATEMP * HTEMP                                           PIP02550
      AAXISQ(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                            PIP02560
   52 CONTINUE                                                          PIP02570
      GO TO 51                                                          PIP02580
   54 CONTINUE                                                          PIP02590
C                                                                       PIP02600
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS       PIP02610
CCC                                                                     PIP02620
CCC      FOR  D2Q(0).                                                   PIP02630
C                                                                       PIP02640
      DO 72 I =  1,6                                                    PIP02650
      READ(5,*) AATEMP                                                  PIP02660
      AATEMP = AATEMP * (DMESH ** 2)                                    PIP02670
      AI6D2Q(I)  = DCMPLX(AATEMP,0.0D0)                                 PIP02680
   72 CONTINUE                                                          PIP02690
C                                                                       PIP02700
CCC      READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP02710
CCC                                                                     PIP02720
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.               PIP02730
CCC                                                                     PIP02740
CCC      BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1) = 2 * H(0) AND      PIP02750
CCC                                                                     PIP02760
CCC      Q(1) = - H(0).                                                 PIP02770
C                                                                       PIP02780
   55 READ(5,49) I,J,K                                                  PIP02790
      IF(I.EQ.9) GO TO 57                                               PIP02800
      HTEMP = DMESH ** I                                                PIP02810
      JJ = 4 - J                                                        PIP02820
      SIGN = 1.0D0                                                      PIP02830
      IF(I.EQ.1) SIGN = -SIGN                                           PIP02840
      IF(I.EQ.3) SIGN = -SIGN                                           PIP02850
      DO 56 KKK = 1,K                                                   PIP02860
      KK = 7 - KKK                                                      PIP02870
      READ(5,*) WATEMP                                                  PIP02880
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      WATEMP = WATEMP * HTEMP * SIGN                                    PIP02890
      AWALLQ(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                            PIP02900
   56 CONTINUE                                                          PIP02910
      GO TO 55                                                          PIP02920
   57 CONTINUE                                                          PIP02930
C                                                                       PIP02940
C        REARRANGE SOME OF THE DATA FOR DQ(R) AND D2Q(R) IN THE         PIP02950
C                                                                       PIP02960
C        AAXISQ AND AWALLQ COEFFICIENT MATRICES.                        PIP02970
C                                                                       PIP02980
      DO 58 I = 1,6                                                     PIP02990
      AAXISQ(2,3,I) = A2DERQ(I + 1)                                     PIP03000
      AAXISQ(1,3,I) = A1DERQ(I + 1)                                     PIP03010
      AWALLQ(2,1,I) = A2DERQ(I)                                         PIP03020
      AWALLQ(1,1,I) = A1DERQ(I)                                         PIP03030
   58 CONTINUE                                                          PIP03040
C                                                                       PIP03050
C        READ IN ALPHA AND REYNOLDS NR                                  PIP03060
C                                                                       PIP03070
      IPLOT = 0                                                         PIP03080
   19 CONTINUE                                                          PIP03090
      READ(5,*) ALPHA,RE                                                PIP03100
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP03110
C                                                                       PIP03120
C*********************************************************************  PIP03130
C                                                                       PIP03140
C                              PART III                                 PIP03150
C                                                                       PIP03160
C        COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS   PIP03170
C                                                                       PIP03180
C        FOR P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES.            PIP03190
C                                                                       PIP03200
C        PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE EQUATIONS.        PIP03210
C                                                                       PIP03220
      AL2 = ALPHA + ALPHA                                               PIP03230
      AL4 = AL2 + AL2                                                   PIP03240
      AL10 = 10.0D0 * ALPHA                                             PIP03250
      ALSQ = ALPHA * ALPHA                                              PIP03260
      AL4TH = ALSQ * ALSQ                                               PIP03270
      ALSQ2 = ALSQ + ALSQ                                               PIP03280
      ALSQ4 = ALSQ2 + ALSQ2                                             PIP03290
      ALSQ5 = ALSQ4 + ALSQ                                              PIP03300
      ALSQ10 = ALSQ5 + ALSQ5                                            PIP03310
      AL3RD2 = ALSQ * ALPHA * 2.0D0                                     PIP03320
C                                                                       PIP03330
C        COMPUTE THE INCREMENTAL PIPE RADIUS AT INTERIOR MESH POINTS.   PIP03340
C                                                                       PIP03350
      DO 60 I = 1,MESH                                                  PIP03360
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```
      R = (DFLOAT(I) - 0.5D0) / DMESH                                   PIP03370
      R2 = R * R                                                        PIP03380
      R3 = R2 * R                                                       PIP03390
      R4 = R3 * R                                                       PIP03400
      RAD(I) = DCMPLX(R,0.0D0)                                          PIP03410
C                                                                       PIP03420
CC       VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)   PIP03430
CCC                                                                     PIP03440
CCC      FIRST EQUATION  FIRST QUADRANT                                 PIP03450
C                                                                       PIP03460
      TEMPR = (1.0D0 + ALSQ * R2) / RE                                  PIP03470
      D2P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03480
      TEMPR = (( 3.0D0/R) + (R*ALSQ5)) / RE                             PIP03490
      D1P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03500
      TEMPR = ALSQ * ( 1.0D0 - ALSQ*R2) / RE                            PIP03510
      TEMPI = - AL2 * (1.0D0 - R2 + (ALSQ*R2) - (ALSQ*R4))              PIP03520
      D0P1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03530
      TEMPR = 1.0D0 + (ALSQ * R2)                                       PIP03540
      E0P1(I) = DCMPLX(TEMPR,0.0D0)                                     PIP03550
C                                                                       PIP03560
CC       V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R) AND D3Q(R)          PIP03570
CCC                                                                     PIP03580
CCC      FIRST EQUATION  SECOND QUADRANT                                PIP03590
C                                                                       PIP03600
      TEMPI = R / RE                                                    PIP03610
      D3Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03620
      TEMPI = 8.0D0 / RE                                                PIP03630
      D2Q1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03640
      TEMPR = AL2 *(R - R3)                                             PIP03650
      TEMPI = ((12.0D0/R) - (ALSQ*R)) / RE                              PIP03660
      D1Q1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03670
      TEMPR = AL2 * (3.0D0 - R2)                                        PIP03680
      TEMPI = - ALSQ5 / RE                                              PIP03690
      D0Q1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03700
      E1Q1(I) = DCMPLX(0.0D0,R)                                         PIP03710
      E0Q1(I) = DCMPLX(0.0D0,3.0D0)                                     PIP03720
C                                                                       PIP03730
CC       VORTICITY TRANSPORT EQN COEFFICIENTS FOR H(0)                  PIP03740
CCC                                                                     PIP03750
CCC      FIRST EQUATION                                                 PIP03760
C                                                                       PIP03770
      TEMPR = AL2 * (2.0D0 - ALSQ + (ALSQ*R2))                          PIP03780
      TEMPI = AL4TH / RE                                                PIP03790
      H0A1(I) = DCMPLX(TEMPR,TEMPI)                                     PIP03800
      TEMPI = -ALSQ                                                     PIP03810
      H0B1(I) = DCMPLX(0.0D0,TEMPI)                                     PIP03820
C                                                                       PIP03830
C        VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)   PIP03840
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C                                                                               PIP03850
CC          SECOND EQUATION  THIRD QUADRANT                                    PIP03860
C                                                                               PIP03870
      TEMPI = -4.0D0 / RE                                                       PIP03880
      D2P2(I) =  DCMPLX(0.0D0,TEMPI)                                            PIP03890
      TEMPI = -12.0D0 / ( RE * R )                                              PIP03900
      D1P2(I) =  DCMPLX(0.0D0,TEMPI)                                            PIP03910
      TEMPR = -AL4 * (1.0D0 - R2)                                               PIP03920
      TEMPI = ALSQ4 / RE                                                        PIP03930
      D0P2(I) =  DCMPLX(TEMPR,TEMPI)                                            PIP03940
      E0P2(I) =  DCMPLX(0.0D0,-2.0D0)                                           PIP03950
C                                                                               PIP03960
CC          V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R), D3Q(R) AND D4Q(R)       PIP03970
CC                                                                              PIP03980
CC          SECOND EQUATION  FOURTH QUADRANT                                    PIP03990
C                                                                               PIP04000
      TEMPR = -R2 / RE                                                          PIP04010
      D4Q2(I) =  DCMPLX(TEMPR,0.0D0)                                            PIP04020
      TEMPR = -(10.0D0 * R) / RE                                                PIP04030
      D3Q2(I) =  DCMPLX(TEMPR,0.0D0)                                            PIP04040
      TEMPR = (-19.0D0 +(ALSQ2 * R2)) / RE                                      PIP04050
      TEMPI = AL2 * (R2 - R4)                                                   PIP04060
      D2Q2(I) =  DCMPLX(TEMPR,TEMPI)                                            PIP04070
      TEMPR = ((3.0D0/R) + (ALSQ10*R)) / RE                                     PIP04080
      TEMPI = AL10 * (R - R3)                                                   PIP04090
      D1Q2(I) =  DCMPLX(TEMPR,TEMPI)                                            PIP04100
      TEMPR = (ALSQ4 -(AL4TH * R2)) / RE                                        PIP04110
      TEMPI = AL2 *((2.0D0 * (1.0D0 - R2))-(ALSQ * (R2 - R4)))                  PIP04120
      D0Q2(I) =  DCMPLX(TEMPR,TEMPI)                                            PIP04130
      TEMPR = -R2                                                               PIP04140
      E2Q2(I) =  DCMPLX(TEMPR,0.0D0)                                            PIP04150
      TEMPR = -5.0D0 * R                                                        PIP04160
      E1Q2(I) =  DCMPLX(TEMPR,0.0D0)                                            PIP04170
      TEMPR = -2.0D0 +(ALSQ * R2)                                               PIP04180
      E0Q2(I) =  DCMPLX(TEMPR,0.0D0)                                            PIP04190
C                                                                               PIP04200
CC          VORTICITY TRANSPORT EQN COEFFICIENTS FOR H(0)                       PIP04210
CC                                                                              PIP04220
CC          SECOND EQUATION                                                     PIP04230
C                                                                               PIP04240
      TEMPR = -AL4TH / RE                                                       PIP04250
      TEMPI = -AL3RD2 * (1.0D0 -(ALPHA * R2))                                   PIP04260
      H0A2(I) =  DCMPLX(TEMPR,TEMPI)                                            PIP04270
      H0B2(I) =  DCMPLX(ALSQ,0.0D0)                                             PIP04280
   60 CONTINUE                                                                  PIP04290
C                                                                               PIP04300
C*****************************************************************************PIP04310
C                                                                               PIP04320
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C                                  PART IV                                      PIP04330
C                                                                               PIP04340
CCCCC      COMPUTATION OF THE A AND B MATRIX ELEMENTS.                          PIP04350
C                                                                               PIP04360
CCCCC      INITIALIZE THE A AND B MATRICES                                      PIP04370
C                                                                               PIP04380
      DO 198 I = 1,100                                                          PIP04390
      DO 198 J = 1,100                                                          PIP04400
      A(I,J) = ZERO                                                             PIP04410
      B(I,J) = ZERO                                                             PIP04420
  198 CONTINUE                                                                  PIP04430
      MH0 = 2 * MESH + 1                                                        PIP04440
      MQ0 = 2 * MESH + 2                                                        PIP04450
C                                                                               PIP04460
CCCCC      A AND B MATRIX ELEMENTS IN TERMS OF P(R)                             PIP04470
C                                                                               PIP04480
CCCCC      FIRST EQUATION  FIRST QUADRANT                                       PIP04490
C                                                                               PIP04500
      DO 200 J = 1,2                                                            PIP04510
      DO 200 K = 1,5                                                            PIP04520
      A0DER = ZERO                                                              PIP04530
      IF (K.EQ.J) A0DER = ONE                                                   PIP04540
      A(J,K) = AAXISP(2,J,K)*D2P1(J) + AAXISP(1,J,K)*D1P1(J) +                  PIP04550
     1         A0DER*D0P1(J)                                                    PIP04560
      B(J,K) = A0DER*E0P1(J)                                                    PIP04570
  200 CONTINUE                                                                  PIP04580
      MP = MESH - 2                                                             PIP04590
      DO 210 J = 3,MP                                                           PIP04600
      DO 210 K = 1,5                                                            PIP04610
      L = J - 3 + K                                                             PIP04620
      A0DER = ZERO                                                              PIP04630
      IF(L.EQ.J) A0DER = ONE                                                    PIP04640
      A(J,L) = A2DERP(K)*D2P1(J) + A1DERP(K)*D1P1(J) + A0DER*D0P1(J)            PIP04650
      B(J,L) = A0DER*E0P1(J)                                                    PIP04660
  210 CONTINUE                                                                  PIP04670
      MMP = MP - 3                                                              PIP04680
      DO 221 J = 1,2                                                            PIP04690
      L = J + MP                                                                PIP04700
      H0D1PA = ZERO                                                             PIP04710
      H0D2PA = ZERO                                                             PIP04720
      DO 220 K = 1,5                                                            PIP04730
      N = MMP + K                                                               PIP04740
      A0DER = ZERO                                                              PIP04750
      IF(N.EQ.L) A0DER = ONE                                                    PIP04760
      A(L,N) = AWALLP(2,J,K)*D2P1(L) + AWALLP(1,J,K)*D1P1(L) +                  PIP04770
     1         A0DER*D0P1(L)                                                    PIP04780
      B(L,N) = A0DER*E0P1(L)                                                    PIP04790
      H0D2PA = H0D2PA + AWALLP(2,J,K)                                           PIP04800
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      HOD1PA = HOD1PA + AWALLP(1,J,K)                                   PIP04810
  220 CONTINUE                                                          PIP04820
      HOD2PA = HOD2PA * D2P1(L)                                         PIP04830
      HOD1PA = HOD1PA * D1P1(L)                                         PIP04840
      HOA1(L) = HOA1(L) - (ONEIMG * (HOD2PA + HOD1PA))                  PIP04850
  221 CONTINUE                                                          PIP04860
C                                                                       PIP04870
CCCCC    A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                       PIP04880
C                                                                       PIP04890
CCCCC    FIRST EQUATION  SECOND QUADRANT                                PIP04900
C                                                                       PIP04910
      DO 251 J = 1,3                                                    PIP04920
      QOD1QA = ZERO                                                     PIP04930
      QOD2QA = ZERO                                                     PIP04940
      QOD3QA = ZERO                                                     PIP04950
      QOD1QB = ZERO                                                     PIP04960
      DO 250 K = 1,6                                                    PIP04970
      KK = K + MESH                                                     PIP04980
      AODER = ZERO                                                      PIP04990
      IF(K.EQ.J) AODER = ONE                                            PIP05000
      A(J,KK) = AAXISQ(3,J,K)*D3Q1(J) + AAXISQ(2,J,K)*D2Q1(J) +         PIP05010
     1          AAXISQ(1,J,K)*D1Q1(J) + AODER*D0Q1(J)                   PIP05020
      B(J,KK) = AAXISQ(1,J,K)*E1Q1(J) + AODER*E0Q1(J)                   PIP05030
      AAXQ1 = AAXISQ(1,J,K)                                             PIP05040
      AAXQ2 = AAXISQ(2,J,K)                                             PIP05050
      IF(J.EQ.3) AAXQ1 = ZERO                                           PIP05060
      IF(J.EQ.3) AAXQ2 = ZERO                                           PIP05070
      QOD1QA = QOD1QA + AAXQ1                                           PIP05080
      QOD2QA = QOD2QA + AAXQ2                                           PIP05090
      QOD3QA = QOD3QA + AAXISQ(3,J,K)                                   PIP05100
      QOD1QB = QOD1QB + AAXQ1                                           PIP05110
  250 CONTINUE                                                          PIP05120
      QOD1QA = QOD1QA * D1Q1(J)                                         PIP05130
      QOD2QA = QOD2QA * D2Q1(J)                                         PIP05140
      QOD3QA = QOD3QA * D3Q1(J)                                         PIP05150
      QOD1QB = QOD1QB * E1Q1(J)                                         PIP05160
      A(J,MQQ) = -(QOD1QA + QOD2QA + QOD3QA)                            PIP05170
      B(J,MQQ) = -QOD1QB                                                PIP05180
  251 CONTINUE                                                          PIP05190
      M = MESH - 3                                                      PIP05200
      DO 260 J = 4,M                                                    PIP05210
      DO 260 K = 1,7                                                    PIP05220
      L = J - 4 + K                                                     PIP05230
      LL = L + MESH                                                     PIP05240
      AODER = ZERO                                                      PIP05250
      IF(L.EQ.J) AODER = ONE                                            PIP05260
      A(J,LL) = A3DERQ(K)*D3Q1(J) + A2DERQ(K)*D2Q1(J) +                 PIP05270
     1          A1DERQ(K)*D1Q1(J) + AODER*D0Q1(J)                       PIP05280
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      B(J,LL) =  A1DERQ(K)*E1Q1(J) + AODER*EOQ1(J)                      PIP05290
  260 CONTINUE                                                          PIP05300
      MM = M - 3                                                        PIP05310
      DO 266 J = 1,3                                                    PIP05320
      L = J + M                                                         PIP05330
      HOD3QA = ZERO                                                     PIP05340
      HOD2QA = ZERO                                                     PIP05350
      HOD1QA = ZERO                                                     PIP05360
      HOD1QB = ZERO                                                     PIP05370
      DO 265 K = 1,6                                                    PIP05380
      KK = 7 - K                                                        PIP05390
      N = MM + K                                                        PIP05400
      NN = N + MESH                                                     PIP05410
      AODER = ZERO                                                      PIP05420
      IF(N.EQ.L)  AODER = ONE                                           PIP05430
      A(L,NN) =  AWALLQ(3,J,K)*D3Q1(L) + AWALLQ(2,J,K)*D2Q1(L) +        PIP05440
     1           AWALLQ(1,J,K)*D1Q1(L) + AODER*DOQ1(L)                  PIP05450
      B(L,NN) =  AWALLQ(1,J,K)*E1Q1(L) + AODER*EOQ1(L)                  PIP05460
      HTEMP = (1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH))          PIP05470
      BETA = DCMPLX(HTEMP,0.0D0)                                        PIP05480
      AWAQ1 = AWALLQ(1,J,K)                                             PIP05490
      AWAQ2 = AWALLQ(2,J,K)                                             PIP05500
      IF(J.EQ.1)  AWAQ1 = ZERO                                          PIP05510
      IF(J.EQ.1)  AWAQ2 = ZERO                                          PIP05520
      HOD1QA = HOD1QA + (AWAQ1 * BETA)                                  PIP05530
      HOD2QA = HOD2QA + (AWAQ2 * BETA)                                  PIP05540
      HOD3QA = HOD3QA + (AWALLQ(3,J,K) * BETA)                          PIP05550
      HOD1QB = HOD1QB + (AWAQ1 * BETA)                                  PIP05560
  265 CONTINUE                                                          PIP05570
      HOD1QA = HOD1QA + TWO                                             PIP05580
      IF(J.EQ.1)  HOD1QA = ZERO                                         PIP05590
      HOD1QB = HOD1QB + TWO                                             PIP05600
      IF(J.EQ.1)  HOD1QB = ZERO                                         PIP05610
      HOD1QA = HOD1QA * D1Q1(L)                                         PIP05620
      HOD2QA = HOD2QA * D2Q1(L)                                         PIP05630
      HOD3QA = HOD3QA * D3Q1(L)                                         PIP05640
      HOD1QB = HOD1QB * E1Q1(L)                                         PIP05650
      HOA1(L) =  HOD1QA + HOD2QA + HOD3QA + HOA1(L)                     PIP05660
      HOB1(L) =  HOD1QB + HOB1(L)                                       PIP05670
  266 CONTINUE                                                          PIP05680
C                                                                       PIP05690
CCC       A AND B MATRIX ELEMENTS IN TERMS OF H(0)                      PIP05700
CCCC                                                                    PIP05710
CC        FIRST EQUATION                                                PIP05720
C                                                                       PIP05730
      DO 267 I = 1,MESH                                                 PIP05740
      A(I,MH0) = HOA1(I)                                                PIP05750
      B(I,MH0) = HOB1(I)                                                PIP05760
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  267 CONTINUE                                                                  PIP05770
C                                                                               PIP05780
CC        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                              PIP05790
CC                                                                              PIP05800
CC        SECOND EQUATION  THIRD QUADRANT                                       PIP05810
C                                                                               PIP05820
      DO 270 J = 1,2                                                            PIP05830
      JJ = J + MESH                                                             PIP05840
      DO 270 K = 1,5                                                            PIP05850
      AODER = ZERO                                                              PIP05860
      IF(K.EQ.J) AODER = ONE                                                    PIP05870
      A(JJ,K) =  AAXISP(2,J,K)*D2P2(J) + AAXISP(1,J,K)*D1P2(J) +                PIP05880
     1           AODER*DOP2(J)                                                  PIP05890
      B(JJ,K) =  AODER*EOP2(J)                                                  PIP05900
  270 CONTINUE                                                                  PIP05910
      DO 275 J = 3,MP                                                           PIP05920
      JJ = J + MESH                                                             PIP05930
      DO 275 K = 1,5                                                            PIP05940
      L = J - 3 + K                                                             PIP05950
      AODER = ZERO                                                              PIP05960
      IF(L.EQ.J) AODER = ONE                                                    PIP05970
      A(JJ,L) =  A2DERP(K)*D2P2(J) + A1DERP(K)*D1P2(J) + AODER*DOP2(J)          PIP05980
      B(JJ,L) =  AODER*EOP2(J)                                                  PIP05990
  275 CONTINUE                                                                  PIP06000
      DO 281 J = 1,2                                                            PIP06010
      L = J + MP                                                                PIP06020
      LL = L + MESH                                                             PIP06030
      HOD1PA = ZERO                                                             PIP06040
      HOD2PA = ZERO                                                             PIP06050
      DO 280 K = 1,5                                                            PIP06060
      N = MMP + K                                                               PIP06070
      AODER = ZERO                                                              PIP06080
      IF (N.EQ.L) AODER = ONE                                                   PIP06090
      A(LL,N) =  AWALLP(2,J,K)*D2P2(L) + AWALLP(1,J,K)*D1P2(L) +                PIP06100
     1           AODER*DOP2(L)                                                  PIP06110
      B(LL,N) =  AODER*EOP2(L)                                                  PIP06120
      HOD1PA = HOD1PA + AWALLP(1,J,K)                                           PIP06130
      HOD2PA = HOD2PA + AWALLP(2,J,K)                                           PIP06140
  280 CONTINUE                                                                  PIP06150
      HOD1PA = HOD1PA * D1P2(L)                                                 PIP06160
      HOD2PA = HOD2PA * D2P2(L)                                                 PIP06170
      HOA2(L) =  HOA2(L) - (ONEIMG * (HOD1PA + HOD2PA))                         PIP06180
  281 CONTINUE                                                                  PIP06190
C                                                                               PIP06200
CC        A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                              PIP06210
CC                                                                              PIP06220
CC        SECOND EQUATION  FOURTH QUADRANT                                      PIP06230
C                                                                               PIP06240
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      DO 286 J = 1,3                                                                                                    PIP06250
      JJ = J + MESH                                                                                                     PIP06260
      QQD1QA = ZERO                                                                                                     PIP06270
      QQD2QA = ZERO                                                                                                     PIP06280
      QQD3QA = ZERO                                                                                                     PIP06290
      QQD4QA = ZERO                                                                                                     PIP06300
      QQD1QB = ZERO                                                                                                     PIP06310
      QQD2QB = ZERO                                                                                                     PIP06320
      DO 285 K = 1,6                                                                                                    PIP06330
      KK = K + MESH                                                                                                     PIP06340
      AODER = ZERO                                                                                                      PIP06350
      IF(K.EQ.J) AODER = ONE                                                                                            PIP06360
      A(JJ,KK) = AAXISQ(4,J,K)*D4Q2(J) + AAXISQ(3,J,K)*D3Q2(J) +                                                        PIP06370
     1           AAXISQ(2,J,K)*D2Q2(J) + AAXISQ(1,J,K)*D1Q2(J) +                                                        PIP06380
     2           AODER*DOQ2(J)                                                                                         PIP06390
      B(JJ,KK) = AAXISQ(2,J,K)*E2Q2(J) + AAXISQ(1,J,K)*E1Q2(J) +                                                        PIP06400
     1           AODER*EOQ2(J)                                                                                         PIP06410
      AAXQ1 = AAXISQ(1,J,K)                                                                                             PIP06420
      AAXQ2 = AAXISQ(2,J,K)                                                                                             PIP06430
      IF(J.EQ.3) AAXQ1 = ZERO                                                                                           PIP06440
      IF(J.EQ.3) AAXQ2 = ZERO                                                                                           PIP06450
      QQD1QA = QQD1QA + AAXQ1                                                                                           PIP06460
      QQD2QA = QQD2QA + AAXQ2                                                                                           PIP06470
      QQD3QA = QQD3QA + AAXISQ(3,J,K)                                                                                   PIP06480
      QQD4QA = QQD4QA + AAXISQ(4,J,K)                                                                                   PIP06490
      QQD1QB = QQD1QB + AAXQ1                                                                                           PIP06500
      QQD2QB = QQD2QB + AAXQ2                                                                                           PIP06510
  285 CONTINUE                                                                                                          PIP06520
      QQD1QA = QQD1QA * D1Q2(J)                                                                                         PIP06530
      QQD2QA = QQD2QA * D2Q2(J)                                                                                         PIP06540
      QQD3QA = QQD3QA * D3Q2(J)                                                                                         PIP06550
      QQD4QA = QQD4QA * D4Q2(J)                                                                                         PIP06560
      QQD1QB = QQD1QB * E1Q2(J)                                                                                         PIP06570
      QQD2QB = QQD2QB * E2Q2(J)                                                                                         PIP06580
      A(JJ,MQQ) = -(QQD1QA + QQD2QA + QQD3QA + QQD4QA)                                                                  PIP06590
      B(JJ,MQQ) = -(QQD1QB + QQD2QB)                                                                                    PIP06600
  286 CONTINUE                                                                                                          PIP06610
      DO 290 J = 4,M                                                                                                    PIP06620
      JJ = J + MESH                                                                                                     PIP06630
      DO 290 K = 1,7                                                                                                    PIP06640
      L = J - 4 + K                                                                                                     PIP06650
      LL = L + MESH                                                                                                     PIP06660
      AODER = ZERO                                                                                                      PIP06670
      IF(L.EQ.J) AODER = ONE                                                                                            PIP06680
      A(JJ,LL) = A4DERQ(K)*D4Q2(J) + A3DERQ(K)*D3Q2(J) +                                                                PIP06690
     1           A2DERQ(K)*D2Q2(J) + A1DERQ(K)*D1Q2(J) + AODER*DOQ2(J)                                                  PIP06700
      B(JJ,LL) = A2DERQ(K)*E2Q2(J) + A1DERQ(K)*E1Q2(J) + AODER*EOQ2(J)                                                  PIP06710
  290 CONTINUE                                                                                                          PIP06720
```

```
      DO 296 J = 1,3                                                    PIP06730
      L = J + M                                                         PIP06740
      LL = L + MESH                                                     PIP06750
      HOD1QA = ZERO                                                     PIP06760
      HOD2QA = ZERO                                                     PIP06770
      HOD3QA = ZERO                                                     PIP06780
      HOD4QA = ZERO                                                     PIP06790
      HOD1QB = ZERO                                                     PIP06800
      HOD2QB = ZERO                                                     PIP06810
      DO 295 K = 1,6                                                    PIP06820
      KK = 7 - K                                                        PIP06830
      N = MM + K                                                        PIP06840
      NN = N + MESH                                                     PIP06850
      AODER = ZERO                                                      PIP06860
      IF(N.EQ.L) AODER = ONE                                            PIP06870
      A(LL,NN) = AWALLQ(4,J,K)*D4Q2(L) + AWALLQ(3,J,K)*D3Q2(L) +        PIP06880
     1           AWALLQ(2,J,K)*D2Q2(L) + AWALLQ(1,J,K)*D1Q2(L) +        PIP06890
     2           AODER*DOQ2(L)                                          PIP06900
      B(LL,NN) = AWALLQ(2,J,K)*E2Q2(L) + AWALLQ(1,J,K)*E1Q2(L) +        PIP06910
     1           AODER*EOQ2(L)                                          PIP06920
      HTEMP = (1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH))          PIP06930
      BETA = DCMPLX(HTEMP,0.0D0)                                        PIP06940
      AWAQ1 = AWALLQ(1,J,K)                                             PIP06950
      AWAQ2 = AWALLQ(2,J,K)                                             PIP06960
      IF(J.EQ.1) AWAQ1 = ZERO                                           PIP06970
      IF(J.EQ.1) AWAQ2 = ZERO                                           PIP06980
      HOD1QA = HOD1QA + (AWAQ1 * BETA)                                  PIP06990
      HOD2QA = HOD2QA + (AWAQ2 * BETA)                                  PIP07000
      HOD3QA = HOD3QA + (AWALLQ(3,J,K) * BETA)                          PIP07010
      HOD4QA = HOD4QA + (AWALLQ(4,J,K) * BETA)                          PIP07020
      HOD1QB = HOD1QB + (AWAQ1 * BETA)                                  PIP07030
      HOD2QB = HOD2QB + (AWAQ2 * BETA)                                  PIP07040
  295 CONTINUE                                                          PIP07050
      HOD1QA = HOD1QA + TWO                                             PIP07060
      HOD1QB = HOD1QB + TWO                                             PIP07070
      IF(J.EQ.1) HOD1QA = ZERO                                          PIP07080
      IF(J.EQ.1) HOD1QB = ZERO                                          PIP07090
      HOD1QA = HOD1QA * D1Q2(L)                                         PIP07100
      HOD2QA = HOD2QA * D2Q2(L)                                         PIP07110
      HOD3QA = HOD3QA * D3Q2(L)                                         PIP07120
      HOD4QA = HOD4QA * D4Q2(L)                                         PIP07130
      HOD1QB = HOD1QB * E1Q2(L)                                         PIP07140
      HOD2QB = HOD2QB * E2Q2(L)                                         PIP07150
      HOA2(L) = HOA2(L) + HOD1QA + HOD2QA + HOD3QA + HOD4QA             PIP07160
      HOB2(L) = HOB2(L) + HOD1QB + HOD2QB                               PIP07170
  296 CONTINUE                                                          PIP07180
C                                                                       PIP07190
C       A AND B MATRIX ELEMENTS IN TERMS OF H(0)                        PIP07200
```

```
C                                                                       PIP07210
CC          SECOND EQUATION                                             PIP07220
C                                                                       PIP07230
      DO 297 I = 1,MESH                                                 PIP07240
      II = I + MESH                                                     PIP07250
      A(II,MH0)  = H0A2(I)                                              PIP07260
      B(II,MH0)  = H0B2(I)                                              PIP07270
  297 CONTINUE                                                          PIP07280
C                                                                       PIP07290
CCC         A AND B MATRIX ELEMENTS FOR THE TWO SPECIAL BOUNDARY        PIP07300
CCCC        CONDITION EQUATIONS INVOLVING GAMMA.                        PIP07310
CCCCC                                                                   PIP07320
CCC         FIRST EQUATION COEFFICIENTS                                 PIP07330
C                                                                       PIP07340
      TEMPR = 4.0D0 / RE                                                PIP07350
      D2P01 = DCMPLX(TEMPR,0.0D0)                                       PIP07360
      TEMPI = 20.0D0 / RE                                               PIP07370
      D2Q01 = DCMPLX(0.0D0,TEMPI)                                       PIP07380
      TEMPR = ALSQ / RE                                                 PIP07390
      TEMPI = - AL2                                                     PIP07400
      D0P01 = DCMPLX(TEMPR,TEMPI)                                       PIP07410
      TEMPR = 6.0D0 * ALPHA                                             PIP07420
      TEMPI = - ALSQ5 / RE                                              PIP07430
      D0Q01 = DCMPLX(TEMPR,TEMPI)                                       PIP07440
      TEMPR = AL4 - AL3RD2                                              PIP07450
      TEMPI = AL4TH / RE                                                PIP07460
      D0H01 = DCMPLX(TEMPR,TEMPI)                                       PIP07470
      E0P01 = ONE                                                       PIP07480
      E0Q01 = DCMPLX(0.0D0,3.0D0)                                       PIP07490
      TEMPI = - ALSQ                                                    PIP07500
      E0H01 = DCMPLX(0.0D0,TEMPI)                                       PIP07510
C                                                                       PIP07520
CC          SECOND EQUATION COEFFICIENTS                                PIP07530
C                                                                       PIP07540
      TEMPI = - 16.0D0 / RE                                             PIP07550
      D2P02 = DCMPLX(0.0D0,TEMPI)                                       PIP07560
      D2Q02 = DCMPLX(TEMPI,0.0D0)                                       PIP07570
      TEMPR = -AL4                                                      PIP07580
      TEMPI = ALSQ4 / RE                                                PIP07590
      D0P02 = DCMPLX(TEMPR,TEMPI)                                       PIP07600
      D0Q02 = DCMPLX(TEMPI,AL4)                                         PIP07610
      TEMPR = - AL4TH / RE                                              PIP07620
      TEMPI = - AL3RD2                                                  PIP07630
      D0H02 = DCMPLX(TEMPR,TEMPI)                                       PIP07640
      E0P02 = DCMPLX(0.0D0,-2.0D0)                                      PIP07650
      E0Q02 = DCMPLX(-2.0D0,0.0D0)                                      PIP07660
      E0H02 = DCMPLX(ALSQ,0.0D0)                                        PIP07670
                                                                        PIP07680
```

```
C                                                                       PIP07690
CC        A AND B MATRIX ELEMENTS RESULTING FROM THE TWO EQUATIONS      PIP07700
C                                                                       PIP07710
      QOTEM1 = ZERO                                                     PIP07720
      QOTEM2 = ZERO                                                     PIP07730
      DO 300 I = 1,6                                                    PIP07740
      A(MHO,I) = D2P01 * AI5D2P(I) + D0P01 * AI5P(I)                    PIP07750
      B(MHO,I) = E0P01 * AI5P(I)                                        PIP07760
      A(MQO,I) = D2P02 * AI5D2P(I) + D0P02 * AI5P(I)                    PIP07770
      B(MQO,I) = E0P02 * AI5P(I)                                        PIP07780
      II = I + MESH                                                     PIP07790
      A(MHO,II) = D2Q01 * AI6D2Q(I)                                     PIP07800
      QOTEM1 = QOTEM1 + AI5D2Q(I)                                       PIP07810
      A(MQO,II) = D2Q02 * AI6D2Q(I)                                     PIP07820
      QOTEM2 = QOTEM2 + AI5D2Q(I)                                       PIP07830
  300 CONTINUE                                                          PIP07840
      A(MHO,MHO) = D0H01                                                PIP07850
      B(MHO,MHO) = E0H01                                                PIP07860
      A(MQO,MHO) = D0H02                                                PIP07870
      B(MQO,MHO) = E0H02                                                PIP07880
      A(MHO,MQO) = D0Q01 - ( QOTEM1 * D2Q01)                            PIP07890
      B(MHO,MQO) = E0Q01                                                PIP07900
      A(MQO,MQO) = D0Q02 - ( QOTEM2 * D2Q02)                            PIP07910
      B(MQO,MQO) = E0Q02                                                PIP07920
C                                                                       PIP07930
CC*****************************************************************   PIP07940
C                                                                       PIP07950
C                            PART V                                     PIP07960
C                                                                       PIP07970
C        COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS         PIP07980
C        USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT        PIP07990
C        EQUATIONS IN THE FORM  A * X = GAMMA * B * X.                  PIP08000
C                                                                       PIP08010
C                                                                       PIP08020
C                                                                       PIP08030
      DO 140 I = 1,100                                                  PIP08040
      EIGVAL(I) = ZERO                                                  PIP08050
      EIGA(I) = ZERO                                                    PIP08060
      EIGB(I) = ZERO                                                    PIP08070
      DO 140 J = 1,100                                                  PIP08080
      EIGVEC(I,J) = ZERO                                                PIP08090
      AA(I,J) = A(I,J)                                                  PIP08100
      BB(I,J) = B(I,J)                                                  PIP08110
  140 CONTINUE                                                          PIP08120
      WRITE(6,145)                                                      PIP08130
  145 FORMAT(1X,//,1X,'**********************************',             PIP08140
     1   '**************************************************',         PIP08150
     2   '**************************************************',//)      PIP08160
```

```
      MESH2 = (MESH * 2) + 2                                                    PIP08170
      IJOB = 1                                                                  PIP08180
      CALL EIGZC (AA,100,BB,100,MESH2,IJOB,EIGA,EIGB,                           PIP08190
     1EIGVEC,100,WK,INEER,IER)                                                  PIP08200
      IF(INEER.EQ.0) WRITE(6,150)                                               PIP08210
  150 FORMAT(1X,///,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',                 PIP08220
     1       'ITERATIONS',//)                                                   PIP08230
      IF(INEER.NE.0) WRITE(6,155) INEER                                         PIP08240
  155 FORMAT(1X,///,10X,'FAILED TO CONVERGE ON THE',I4,1X,                      PIP08250
     1       'TH EIGENVALUE',//)                                                PIP08260
      DO 165 I = 1,MESH2                                                        PIP08270
      EIGVAL(I) = EIGA(I)/EIGB(I)                                               PIP08280
      GAMR(I) = DREAL(EIGVAL(I))                                                PIP08290
  165 CONTINUE                                                                  PIP08300
      NGAM = 1                                                                  PIP08310
      GAMMAX = GAMR(1)                                                          PIP08320
      DO 170 I = 2,MESH2                                                        PIP08330
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                           PIP08340
      GAMMAX = GAMR(I)                                                          PIP08350
      NGAM = I                                                                  PIP08360
  169 CONTINUE                                                                  PIP08370
  170 CONTINUE                                                                  PIP08380
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                                    PIP08390
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,          PIP08400
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)              PIP08410
C                                                                               PIP08420
C*****************************************************************            PIP08430
CCC                                                                             PIP08440
CCC                            PART VI                                          PIP08450
CCC                                                                             PIP08460
CCC        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING            PIP08470
CCC        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL               PIP08480
CCC        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.                     PIP08490
CCC                                                                             PIP08500
C                                                                               PIP08510
      DO 351 I = 1,MESH2                                                        PIP08520
      SUMA = ZERO                                                               PIP08530
      SUMB = ZERO                                                               PIP08540
      DO 350 J = 1,MESH2                                                        PIP08550
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                                 PIP08560
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                                 PIP08570
  350 CONTINUE                                                                  PIP08580
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                                 PIP08590
  351 CONTINUE                                                                  PIP08600
      WRITE(6,375)                                                              PIP08610
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)                   PIP08620
      WRITE(6,380) (ERROR(I),I = 1,MESH2)                                       PIP08630
                                                                                PIP08640
```

```
  380 FORMAT(1X,2D30.18)                                                    PIP08650
C                                                                           PIP08660
C*************************************************************************  PIP08670
C                                                                           PIP08680
C                                PART VII                                   PIP08690
C                                                                           PIP08700
C       COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).                   PIP08710
C                                                                           PIP08720
C       OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE            PIP08730
C                                                                           PIP08740
C       EIGENVALUE,  GAMMA*.                                                PIP08750
C                                                                           PIP08760
      DO 400 I = 1,MESH                                                     PIP08770
      II = I + MESH                                                         PIP08780
      P(I) = EIGVEC(I,NGAM)                                                 PIP08790
      Q(I) = EIGVEC(II,NGAM)                                                PIP08800
  400 CONTINUE                                                              PIP08810
      H0 = EIGVEC(MH0,NGAM)                                                 PIP08820
      Q0 = EIGVEC(MQ0,NGAM)                                                 PIP08830
      WRITE(6,401) EIGVAL(NGAM)                                             PIP08840
  401 FORMAT(1X,///,5X,'EIGENVECTOR CORRESPONDING TO',                      PIP08850
     1       ' THE LEAST STABLE EIGENVALUE, GAMMA* =',2D27.18,/)            PIP08860
      DO 402 I = 1,MESH                                                     PIP08870
      II = I + MESH                                                         PIP08880
      WRITE(6,404) I,P(I),II,Q(I)                                           PIP08890
  404 FORMAT(2X,I2,2D28.18,2X,I2,2D23.18)                                   PIP08900
  402 CONTINUE                                                              PIP08910
      WRITE(6,403) H0,Q0                                                    PIP08920
  403 FORMAT(1X,/,2X,'H0',2D28.18,2X,'Q0',2D28.18,//)                       PIP08930
C                                                                           PIP08940
C       COMPUTATION OF DQ(R)                                                PIP08950
C                                                                           PIP08960
      DO 411 J = 1,3                                                        PIP08970
      DQ(J) = ZERO                                                          PIP08980
      QOD1QA = ZERO                                                         PIP08990
      DO 410 K = 1,6                                                        PIP09000
      AAXQ1 = AAXISQ(1,J,K)                                                 PIP09010
      IF(J.EQ.3) AAXQ1 = ZERO                                               PIP09020
      DQ(J) = DQ(J) + Q(K) * AAXISQ(1,J,K)                                  PIP09030
      QOD1QA = QOD1QA + AAXQ1                                               PIP09040
  410 CONTINUE                                                              PIP09050
      DQ(J) = DQ(J) - (QOD1QA * Q0)                                         PIP09060
  411 CONTINUE                                                              PIP09070
C                                                                           PIP09080
      DO 420 J = 4,M                                                        PIP09090
      DQ(J) = ZERO                                                          PIP09100
      DO 420 K = 1,7                                                        PIP09110
      L = J - 4 + K                                                         PIP09120
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      DQ(J) = DQ(J) +(Q(L) * A1DERQ(K))                                  PIP09130
  420 CONTINUE                                                           PIP09140
C                                                                        PIP09150
      DO 431 J = 1,3                                                     PIP09160
      L = J + M                                                          PIP09170
      DQ(L) = ZERO                                                       PIP09180
      HOD1QA = ZERO                                                      PIP09190
      DO 430 K = 1,6                                                     PIP09200
      AWAQ1 = AWALLQ(1,J,K)                                              PIP09210
      IF(J.EQ.1) AWAQ1 = ZERO                                            PIP09220
      KK = 7 - K                                                         PIP09230
      N = MM + K                                                         PIP09240
      HTEMP =  1.0D0 - ((2.0D0 * DFLOAT(KK) - 1.0D0) / DMESH)            PIP09250
      BETA = DCMPLX(HTEMP,0.0D0)                                         PIP09260
      HOD1QA = HOD1QA + (AWAQ1 * BETA)                                   PIP09270
      DQ(L) = DQ(L) +(Q(N) * AWALLQ(1,J,K))                              PIP09280
  430 CONTINUE                                                           PIP09290
      HOD1QA = HOD1QA + TWO                                              PIP09300
      IF(J.EQ.1) HOD1QA = ZERO                                           PIP09310
      DQ(L) = DQ(L) + (HOD1QA * HO)                                      PIP09320
  431 CONTINUE                                                           PIP09330
C                                                                        PIP09340
      DO 440 I = 1,MESH                                                  PIP09350
      U(I) = RAD(I) * ((THREE*Q(I)) + (RAD(I)*DQ(I)) - (ONEIMG*P(I)))    PIP09360
  440 CONTINUE                                                           PIP09370
CC                                                                       PIP09380
CC        REARRANGE DATA FOR PLOT ROUTINE                                PIP09390
C                                                                        PIP09400
      M2 = MESH + 2                                                      PIP09410
      DO 450 I = 1,MESH                                                  PIP09420
      II = M2 - I                                                        PIP09430
      III = II - 1                                                       PIP09440
      U(II) = U(III)                                                     PIP09450
      RAD(II) = RAD(III)                                                 PIP09460
  450 CONTINUE                                                           PIP09470
      RAD(1) = ZERO                                                      PIP09480
      RAD(M2) = ONE                                                      PIP09490
      U(1) = ZERO                                                        PIP09500
      U(M2) = ZERO                                                       PIP09510
      DO 460 I = 1,M2                                                    PIP09520
      RADR(I) = DREAL(RAD(I))                                            PIP09530
  460 CONTINUE                                                           PIP09540
C                                                                        PIP09550
CCC       COMPUTE THE NORMALIZED PERTURBATION VELOCITY AND               PIP09560
CCCC                                                                     PIP09570
CC        NORMALIZED AMPLITUDE AND OUTPUT THE RESULTS.                   PIP09580
C                                                                        PIP09590
      DO 465 I = 1,M2                                                    PIP09600
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AD-A126 767 A NUMERICAL ANALYSIS OF PIPE FLOW STABILITY(U) NAVAL POSTGRADUATE SCHOOL MONTEREY CA D B WALLACE DEC 82 3/3

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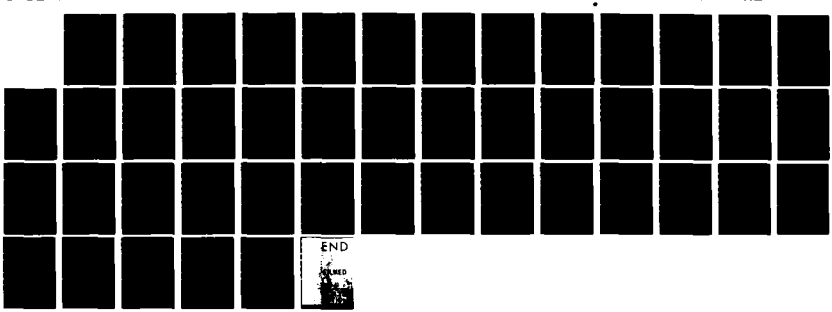

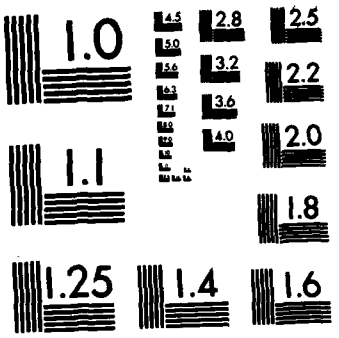

MICROCOPY RESOLUTION TEST CHART
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      UAMP(I) = CDABS(U(I))                                                    PIP09610
  465 CONTINUE                                                                 PIP09620
      AMPMAX = UAMP(1)                                                         PIP09630
      IMAX = 1                                                                 PIP09640
      DO 470 J = 2,M2                                                          PIP09650
      AMTEMP = UAMP(J)                                                         PIP09660
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                           PIP09670
      AMPMAX = UAMP(J)                                                         PIP09680
      IMAX = J                                                                 PIP09690
  471 CONTINUE                                                                 PIP09700
  470 CONTINUE                                                                 PIP09710
      WRITE(6,472)                                                             PIP09720
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',                    PIP09730
     1 24X,'NORMALIZED AMPLITUDE',/)                                           PIP09740
      DO 475 K = 1,M2                                                          PIP09750
      UAMNOR(K) = UAMP(K) / AMPMAX                                             PIP09760
      AMPNOR(K) = UAMNOR(K)                                                    PIP09770
      UNORM(K) = U(K) / U(IMAX)                                                PIP09780
      URDP(K) = DREAL(UNORM(K))                                                PIP09790
      UR(K) = URDP(K)                                                          PIP09800
      UIDP(K) = DIMAG(UNORM(K))                                                PIP09810
      UI(K) = UIDP(K)                                                          PIP09820
      KK = K - 1                                                               PIP09830
      WRITE(6,466) KK,URDP(K),UIDP(K),UAMNOR(K)                                PIP09840
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                       PIP09850
  475 CONTINUE                                                                 PIP09860
C                                                                              PIP09870
CC*****************************************************************          PIP09880
CCC                                                                            PIP09890
CCCC                          PART VIII                                        PIP09900
CCCCC                                                                          PIP09910
CC        PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.            PIP09920
C                                                                              PIP09930
      IPLOT = IPLOT + 1                                                        PIP09940
      RMESH = FLOAT(MESH)                                                      PIP09950
      ALPLOT = ALPHA                                                           PIP09960
      REPLOT = RE                                                              PIP09970
      GAMPLT = GAMMAX                                                          PIP09980
      CALL PLOTG(RADR,UR,M2,1,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)        PIP09990
      CALL PLOTG(RADR,UI,M2,2,1,5,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)        PIP10000
C     CALL PLOTG(RADR,AMPNOR,M2,3,1,2,' ',1,' ',1,0.1,-1.1,1.5,5.)             PIP10010
      CALL PLOTG(X1,Y1,2,3,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)           PIP10020
      XO = 0.15                                                                PIP10030
      YO = 5.6                                                                 PIP10040
      IF(IPLOT.EQ.4) IPLOT = 1                                                 PIP10050
      IF(IPLOT.EQ.2) YO = 12.1                                                 PIP10060
      IF(IPLOT.EQ.3) YO = 18.6                                                 PIP10070
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                            PIP10080
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      GO TO 19                                                          PIP10090
  180 CONTINUE                                                          PIP10100
      CALL PLOT(0.0,0.0,999)                                            PIP10110
      STOP                                                              PIP10120
      END                                                               PIP10130
C                                                                       PIP10140
CC.....................................................................PIP10150
CC                                                                      PIP10160
CC        SUBROUTINE TITLE1                                             PIP10170
CC                                                                      PIP10180
CC        SUBROUTINE TO WRITE TITLE ON PLOTS                            PIP10190
C                                                                       PIP10200
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)               PIP10210
      RN = 1.0                                                          PIP10220
      YHT = 0.14                                                        PIP10230
      CALL NEWPEN(2)                                                    PIP10240
      CALL SYMBOL(XO,YO,YHT,                                            PIP10250
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)              PIP10260
      YHT = 0.08                                                        PIP10270
      DELY = 0.20                                                       PIP10280
      CALL NEWPEN(1)                                                    PIP10290
      YO = YO - 0.8                                                     PIP10300
      XO = XO + 2.5                                                     PIP10310
      XN = XO + .75                                                     PIP10320
      CALL SYMBOL(XO,YO,YHT,'N      = ',0.0,9)                          PIP10330
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                  PIP10340
      YO = YO - DELY                                                    PIP10350
      CALL SYMBOL(XO,YO,YHT,'MESH   = ',0.0,9)                          PIP10360
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                               PIP10370
      YO = YO - DELY                                                    PIP10380
      CALL SYMBOL(XO,YO,YHT,'ALPHA  = ',0.0,9)                          PIP10390
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                               PIP10400
      YO = YO - DELY                                                    PIP10410
      CALL SYMBOL(XO,YO,YHT,'REY NR = ',0.0,9)                          PIP10420
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                               PIP10430
      YO = YO - DELY                                                    PIP10440
      CALL SYMBOL(XO,YO,YHT,'GAMMA* = ',0.0,9)                          PIP10450
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                               PIP10460
      YO = YO - 3.4                                                     PIP10470
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                 PIP10480
      YO = YO - DELY                                                    PIP10490
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                 PIP10500
      YO = YO - DELY                                                    PIP10510
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                 PIP10520
CC                                                                      PIP10530
CC        WRITE THE AXIS LABLES                                         PIP10540
C                                                                       PIP10550
      XO = XO - 2.55                                                    PIP10560
```

```
      YO = YO + 0.3                                                              PIP10570
      YHT = 0.12                                                                 PIP10580
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)           PIP10590
      XO = XO + 2.8                                                              PIP10600
      YO = YO - 1.0                                                              PIP10610
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                                PIP10620
      RETURN                                                                     PIP10630
      END                                                                        PIP10640
/*                                                                               PIP10650
//GO.SYSIN DD *                                                                  PIP10660
2 1 5                                                                            PIP10670
     -0.75414446417998842 7D+00                                                  PIP10680
      0.37221235445036506 0D+00                                                  PIP10690
      0.61900532859680124 1D+00                                                  PIP10700
     -0.28537596210775519 1D+00                                                  PIP10710
      0.48302743240576178 3D-01                                                  PIP10720
2 2 5                                                                            PIP10730
      0.12837477797513325 9D+01                                                  PIP10740
     -0.25705545687783668 6D+01                                                  PIP10750
      0.13880994671403148 5D+01                                                  PIP10760
     -0.10479573712255696 4D+00                                                  PIP10770
      0.35030590927518066 D-02                                                   PIP10780
1 1 4                                                                            PIP10790
     -0.10265151515151515 138D+01                                                PIP10800
      0.13977272727272727 271D+01                                                PIP10810
     -0.44318181818181818 135D+00                                                PIP10820
      0.71969696969696947 4D-01                                                  PIP10830
1 2 4                                                                            PIP10840
     -0.60227272727272640 5D+00                                                  PIP10850
      0.34090909090907173 3D-01                                                  PIP10860
      0.64772727272727337 3D+00                                                  PIP10870
     -0.79545454545454447 0D-01                                                  PIP10880
9 0 0                                                                            PIP10890
      0.13769010879218464 0D+01                                                  PIP10900
     -0.61195603907637506 6D+00                                                  PIP10910
      0.33045626110124208 5D+00                                                  PIP10920
     -0.11240008880994618 9D+00                                                  PIP10930
      0.16998778863232564 4D-01                                                  PIP10940
     -0.44265097690941326 3D+01                                                  PIP10950
      0.78006710084862640 5D+01                                                  PIP10960
     -0.48423623445825754 2D+01                                                  PIP10970
      0.17388987566607383 4D+01                                                  PIP10980
     -0.27069765147029635 5D+00                                                  PIP10990
2 1 5                                                                            PIP11000
     -0.54166666666666580 8D+01                                                  PIP11010
      0.24444444444444417 6D+01                                                  PIP11020
     -0.49999999999999483 7D+00                                                  PIP11030
      0.95238095238094885 9D-01                                                  PIP11040
```

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        -0.92592592592588987 9D-02                                      PIP11050
2 2 5                                                                   PIP11060
         0.17499999999999992 45D+01                                     PIP11070
        -0.27777777777777759 0D+01                                      PIP11080
         0.14999999999999917 8D+01                                      PIP11090
        -0.14285714285713968 5D+00                                      PIP11100
         0.92592592592587591 4D-02                                      PIP11110
1 1 4                                                                   PIP11120
         0.16666666666666586 5D+00                                      PIP11130
         0.10000000000000000 22D+01                                     PIP11140
        -0.29999999999999989 2D+00                                      PIP11150
         0.47619047619047633 8D-01                                      PIP11160
1 2 4                                                                   PIP11170
        -0.99999999999999733 5D+00                                      PIP11180
         0.16666666666666429 8D+00                                      PIP11190
         0.60000000000000011 99D+00                                     PIP11200
        -0.71428571428571285 8D-01                                      PIP11210
9 0 0                                                                   PIP11220
4 1 6                                                                   PIP11230
         0.11500000000000007 14D+03                                     PIP11240
        -0.38444444444444502 55D+02                                     PIP11250
         0.16400000000000038 58D+02                                     PIP11260
        -0.52244897959184979 40D+01                                     PIP11270
         0.11234567901234950 9D+01                                      PIP11280
        -0.11570247933884755 8D+00                                      PIP11290
4 2 6                                                                   PIP11300
        -0.26000000000000800 29D+02                                     PIP11310
         0.17666666666667016 0D+02                                      PIP11320
        -0.11200000000000023 02D+02                                     PIP11330
         0.39591836734695107 2D+01                                      PIP11340
        -0.66666666666670049 1D+00                                      PIP11350
         0.57851239669425180 7D-01                                      PIP11360
4 3 6                                                                   PIP11370
         0.10000000000000852 65D+01                                     PIP11380
        -0.62222222222228040 8D+01                                      PIP11390
         0.92000000000000457 59D+01                                     PIP11400
        -0.64489795918368990 5D+01                                      PIP11410
         0.19876543209876977 9D+01                                      PIP11420
        -0.16528925619835255 2D+00                                      PIP11430
3 1 5                                                                   PIP11440
        -0.17499999999999975 1D+02                                      PIP11450
         0.11111111111110760 8D+00                                      PIP11460
         0.12000000000000013 72D+01                                     PIP11470
        -0.34693877551020739 5D+00                                      PIP11480
         0.43209876543210391 1D-01                                      PIP11490
3 2 5                                                                   PIP11500
         0.13499999999999968 0D+02                                      PIP11510
        -0.38888888888888821 3D+01                                      PIP11520
```

```
      -0.14210854715202003 7D-13
       0.55102040816326658 4D+00
      -0.67901234567901785 9D-01
3 3 5
      -0.15500000000000113 7D+02
       0.98888888888889106 0D+01
      -0.60000000000000532 9D+01
       0.14489795918367498 5D+01
       0.67901234567899093 5D-01
2 1 4
      -0.46666666666666658 5D+01
       0.21111111111111102 7D+01
      -0.31999999999999993 0D+00
       0.34013605442176852 1D-01
2 2 4
       0.99999999999997868 4D+00
      -0.24444444444444493 0D+01
       0.13200000000000011 7D+01
      -0.81632653061224511 0D-01
1 1 3
       0.24999999999999988 9D+01
       0.22222222222222293 0D+00
      -0.19999999999999978 8D-01
1 2 3
      -0.45000000000000035 5D+01
       0.13333333333333304 0D+01
       0.18000000000000007 0D+00
9 9 0
       0.21656250000000071 1D+02
      -0.40104166666667335 8D+01
       0.17325000000000434 5D+01
      -0.63137755102042347 8D+00
       0.14853395061728838 5D+00
      -0.16270661157025321 2D-01
4 1 6
       0.11500000000000071 4D+03
      -0.38444444444445025 5D+02
       0.16400000000000385 8D+02
      -0.52244897959184979 4D+01
       0.11234567901234950 9D+01
      -0.11570247933884755 8D+00
4 2 6
      -0.26000000000000802 9D+02
       0.17666666666667016 0D+02
      -0.11200000000000230 2D+02
       0.39591836734695107 2D+01
      -0.66666666666670049 1D+00
       0.57851239669425180 7D-01
```

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PIP11530
PIP11540
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PIP11940
PIP11950
PIP11960
PIP11970
PIP11980
PIP11990
PIP12000
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4 3 6                                                                   PIP12010
      0.10000000000000852650D+01                                        PIP12020
     -0.62222222222222804080D+01                                        PIP12030
      0.92000000000000457590D+01                                        PIP12040
     -0.64489795918368990500D+01                                        PIP12050
      0.19876543209876977900D+01                                        PIP12060
     -0.16528925619835255200D+00                                        PIP12070
3 1 5                                                                   PIP12080
     -0.17499999999999997510D+02                                        PIP12090
      0.11111111111111076080D+00                                        PIP12100
      0.12000000000000013720D+01                                        PIP12110
     -0.34693877551020739500D+00                                        PIP12120
      0.43209876543210391100D-01                                        PIP12130
3 2 5                                                                   PIP12140
      0.13499999999999996800D+02                                        PIP12150
     -0.38888888888888821300D+01                                        PIP12160
     -0.14210854715202003700D-13                                        PIP12170
      0.55102040816326658400D+00                                        PIP12180
     -0.67901234567901785900D-01                                        PIP12190
3 3 5                                                                   PIP12200
     -0.15500000000000011370D+02                                        PIP12210
      0.98888888888889106000D+01                                        PIP12220
     -0.60000000000000053290D+01                                        PIP12230
      0.14489795918367498500D+01                                        PIP12240
      0.67901234567899093500D-01                                        PIP12250
2 1 4                                                                   PIP12260
     -0.46666666666666658500D+01                                        PIP12270
      0.21111111111111102700D+01                                        PIP12280
     -0.31999999999999993000D+00                                        PIP12290
      0.34013605442176852100D-01                                        PIP12300
2 2 4                                                                   PIP12310
      0.99999999999997868400D+00                                        PIP12320
     -0.24444444444444493000D+01                                        PIP12330
      0.13200000000000011700D+01                                        PIP12340
     -0.81632653061224511000D-01                                        PIP12350
1 1 3                                                                   PIP12360
      0.24999999999999988300D+01                                        PIP12370
      0.22222222222222229300D+00                                        PIP12380
     -0.19999999999999997300D-01                                        PIP12390
1 2 3                                                                   PIP12400
     -0.45000000000000003550D+01                                        PIP12410
      0.13333333333333330400D+01                                        PIP12420
      0.18000000000000000700D+00                                        PIP12430
9 0 0                                                                   PIP12440
0.0D0      1150.0D0                                                     PIP12450
0.5D0      1150.0D0                                                     PIP12460
1.0D0      1150.0D0                                                     PIP12470
2.0D0      1150.0D0                                                     PIP12480
```

```
4.0D0      1150.0D0                                                  PIP12490
8.0D0      1150.0D0                                                  PIP12500
16.0D0     1150.0D0                                                  PIP12510
32.0D0     1150.0D0                                                  PIP12520
0.0D0      2300.0D0                                                  PIP12530
0.5D0      2300.0D0                                                  PIP12540
1.0D0      2300.0D0                                                  PIP12550
2.0D0      2300.0D0                                                  PIP12560
4.0D0      2300.0D0                                                  PIP12570
8.0D0      2300.0D0                                                  PIP12580
16.0D0     2300.0D0                                                  PIP12590
32.0D0     2300.0D0                                                  PIP12600
-1.0D0  0.0D0                                                        PIP12610
/*                                                                   PIP12620
//                                                                   PIP12630
```

```
//WALLACE JOB (2027,0084),'THESIS N = 6',CLASS=C                    PIP00010
//*MAIN LINES=(20)                                                  PIP00020
//  EXEC FRTXCLGP,IMSL=DP,REGION.GO=2048K                           PIP00030
//FORT.SYSIN DD *                                                   PIP00040
C                                                                   PIP00050
C*******************************************************************PIP00060
C                                                                   PIP00070
C       THIS PROGRAM WAS DEVELOPED TO PERFORM A NUMERICAL ANALYSIS  PIP00080
C       OF THREE DIMENSIONAL PIPE FLOW STABILITY FOR AXIAL WAVE     PIP00090
C       NUMBER N = 6. THE PROGRAM IS DIVIDED INTO EIGHT PARTS.      PIP00100
C       THE FIRST PART COMPUTES THE CENTRAL FINITE DIFFERENCE       PIP00110
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT THE    PIP00120
C       INTERIOR MESH POINTS ALONG THE PIPE RADIUS. IN THE SECOND   PIP00130
C       PART,  PRE-COMPUTED NON-CENTRAL FINITE DIFFERENCE           PIP00140
C       COEFFICIENTS FOR THE DERIVATIVES OF P(R) AND Q(R) AT POINTS PIP00150
C       NEAR THE AXIS AND WALL ARE READ IN AS DATA. THE THIRD PART  PIP00160
C       COMPUTES THE VORTICITY TRANSPORT EQUATION COEFFICIENTS OF   PIP00170
C       P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES AT ALL RADIAL PIP00180
C       MESH POINTS (STATIONS). THE FOURTH PART COMPUTES THE A AND B PIP00190
C       MATRIX ELEMENTS,  WHICH MAKE-UP THE EIGENSYSTEM TO BE SOLVED PIP00200
C       IN THE FORMAT,  A * X = GAMMA * B * X.  IN THE FIFTH PART   PIP00210
C       THE A AND B MATRICES COMPRISING THE EIGENSYSTEM ARE SOLVED  PIP00220
C       IN THE SUBROUTINE EIGZC.  THE RESULTING EIGENVALUES AND     PIP00230
C       CORRESPONDING EIGENVECTOR REPRESENT GAMMA AND THE COLUMN    PIP00240
C       VECTOR X,  RESPECTIVELY, AND ARE SOLUTIONS OF THE GOVERNING PIP00250
C       PARTIAL DIFFERENTIAL EQUATIONS.  THE SIXTH PART VERIFIES    PIP00260
C       THE SOLUTION OF THE PDE AND THE MAGNITUDE OF THE RESIDUAL   PIP00270
C       IS DETERMINED.  THE SEVENTH PART COMPUTES THE NORMALIZED    PIP00280
```

```
      DATA ZERO/(0.0D0,0.0D0)/,ONE/(1.0D0,0.0D0)/,FOUR/(4.0D0,0.0D0)/   PIP00970
      DATA SIXTHS/(0.0D0,5.0D0)/                                        PIP00980
      DATA AAXISQ/72*(0.0D0,0.0D0)/,AWALLQ/72*(0.0D0,0.0D0)/            PIP00990
      DATA A1DERQ/7*(0.0D0,0.0D0)/,A2DERQ/7*(0.0D0,0.0D0)/              PIP01000
      DATA A3DERQ/7*(0.0D0,0.0D0)/,A4DERQ/7*(0.0D0,0.0D0)/              PIP01010
      DATA AAXISP/20*(0.0D0,0.0D0)/,AWALLP/20*(0.0D0,0.0D0)/            PIP01020
      DATA A1DERP/5*(0.0D0,0.0D0)/,A2DERP/5*(0.0D0,0.0D0)/              PIP01030
      DATA X1(1)/0.0/,X1(2)/1.0/,Y1(1)/0.0/,Y1(2)/0.0/                  PIP01040
      DATA MESH/50/                                                     PIP01050
C                                                                       PIP01060
CC*********************************************************************PIP01070
CC                                                                      PIP01080
CC                              PART I                                  PIP01090
CC                                                                      PIP01100
CC       COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FOURTH ORDER       PIP01110
CC                                                                      PIP01120
CC       DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01130
C                                                                       PIP01140
      DMESH = DFLOAT(MESH)                                              PIP01150
      H4TH = DMESH ** 4                                                 PIP01160
      ATEMP = (-1.0D0/6.0D0) * H4TH                                     PIP01170
      A4DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01180
      A4DERQ(7) = A4DERQ(1)                                             PIP01190
      ATEMP = 2.0D0 * H4TH                                              PIP01200
      A4DERQ(2) = DCMPLX(ATEMP,0.0D0)                                   PIP01210
      A4DERQ(6) = A4DERQ(2)                                             PIP01220
      ATEMP = -6.5D0 * H4TH                                             PIP01230
      A4DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01240
      A4DERQ(5) = A4DERQ(3)                                             PIP01250
      ATEMP = (56.0D0/6.0D0) * H4TH                                     PIP01260
      A4DERQ(4) = DCMPLX(ATEMP,0.0D0)                                   PIP01270
C                                                                       PIP01280
CC       COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR THIRD ORDER        PIP01290
CC                                                                      PIP01300
CC       DERIVATIVE OF Q(R) AT INTERIOR MESH POINTS.                    PIP01310
C                                                                       PIP01320
      H3RD = DMESH ** 3                                                 PIP01330
      ATEMP = 0.125D0 * H3RD                                            PIP01340
      A3DERQ(1) = DCMPLX(ATEMP,0.0D0)                                   PIP01350
      A3DERQ(7) = -A3DERQ(1)                                            PIP01360
      A3DERQ(6) = DCMPLX(H3RD,0.0D0)                                    PIP01370
      A3DERQ(2) = -A3DERQ(5)                                            PIP01380
      ATEMP = (13.0D0/8.0D0) * H3RD                                     PIP01390
      A3DERQ(3) = DCMPLX(ATEMP,0.0D0)                                   PIP01400
      A3DERQ(5) = -A3DERQ(3)                                            PIP01410
C                                                                       PIP01420
CC       COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR SECOND ORDER       PIP01430
C                                                                       PIP01440
```

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C                                                                       PIP00490
CCC     AXIAL PERTURBATION VELOCITY U(R).  THE EIGHTH AND LAST          PIP00500
CCC                                                                     PIP00510
CCC     PART PLOTS THE PERTURBATION VELOCITY VERSUS PIPE RADIUS.        PIP00520
CCC                                                                     PIP00530
CCC     REQUIRED SUBROUTINES ARE INCLUDED AT THE END OF THE PROGRAM     PIP00540
CCC                                                                     PIP00550
CCC     LISTING.  THIS IS A GENERALIZED PROGRAM FOR VARIABLE SIZED      PIP00560
CCC                                                                     PIP00570
CCC     MESH UP TO A MAXIMUM MESH SIZE OF 50.  BECAUSE OF THE           PIP00580
CCC                                                                     PIP00590
CCC     LARGE AMOUNT OF MEMORY REQUIRED TO RUN THIS PROGRAM,  IT        PIP00600
CCC                                                                     PIP00610
CCC     CAN ONLY BE RUN IN THE BATCH MODE (MVS).  AVERAGE CPU TIME      PIP00620
CCC                                                                     PIP00630
CCC     REQUIRED FOR ONE SET OF DATA,  I.E. ONE AXIAL WAVE NUMBER       PIP00640
CCC                                                                     PIP00650
CCC     AND ONE REYNOLDS NUMBER,  IS 61 SECONDS.                        PIP00660
CCC*****************************************************************  PIP00670
CCC                                                                     PIP00680
CCC     DEFINE PROGRAM VARIABLES AND ARRAYS.                            PIP00690
C                                                                       PIP00700
      COMPLEX*16 A(100,100),B(100,100)                                  PIP00710
      COMPLEX*16 ZERO,ONE,AAXISQ(4,3,6),AWALLQ(4,3,6),A1DERQ(7)         PIP00720
      COMPLEX*16 A2DERQ(7),A3DERQ(7),A4DERQ(7),A0DER                    PIP00730
      COMPLEX*16 D3Q1(50),D2Q1(50),D1Q1(50),D0Q1(50)                    PIP00740
      COMPLEX*16 D4Q2(50),D3Q2(50),D2Q2(50),D1Q2(50),D0Q2(50)           PIP00750
      COMPLEX*16 E1Q1(50),E0Q1(50),E2Q2(50),E1Q2(50),E0Q2(50)           PIP00760
      COMPLEX*16 AAXISP(2,2,5),AWALLP(2,2,5),A1DERP(5),A2DERP(5)        PIP00770
      COMPLEX*16 D2P1(50),D1P1(50),D0P1(50),D2P2(50),D1P2(50)           PIP00780
      COMPLEX*16 D0P2(50),E0P1(50),E0P2(50)                             PIP00790
      COMPLEX*16 EIGA(100),EIGB(100),EIGVEC(100,100),EIGVAL(100)        PIP00800
      COMPLEX*16 AA(100,100),BB(100,100),SIXIMG,FOUR                    PIP00810
      COMPLEX*16 Q(50),DQ(50),P(50),J(52),RAD(52),UNORM(52)             PIP00820
      COMPLEX*16 ERROR(10),WK(1,1),SUMA,SUMB                            PIP00830
      REAL*8 GAMR(100),GAMMAX,DMESH                                     PIP00840
      REAL*8 H4TH,H3RD,H2ND,HIST,HTEMP,ATEMP,AATEMP,WATEMP,R,RE,ALPHA   PIP00850
      REAL*8 TEMPR,TEMPI,SIGN,R2,R3,R4,R5                               PIP00860
      REAL*8 AL2,AL12,AL14,AL24,ALSQ,ALSQ2,ALSQ6,ALSQ9,ALSQ24,ALSQ36    PIP00870
      REAL*8 ALSQ57,AL4TH,AL3RD2                                        PIP00880
      REAL*8 UAMP(52),AMPMAX,AMTEMP,JAMNOR(52),URDP(52),UIDP(52)        PIP00890
      REAL*4 RADR(52),UR(52),UI(52),AMPNOR(52)                          PIP00900
      REAL*4 X1(2),Y1(2),ALPLOT,REPLOT,GAMPLT,RMESH                     PIP00910
      INTEGER MESH,M,MM,M2,IMAX,MP,M1P,MESH2,NGAM                       PIP00920
C                                                                       PIP00930
CCC     INITIALIZE THE COEFFICIENT ARRAYS AND SEVERAL CONSTANTS         PIP00940
C                                                                       PIP00950
                                                                        PIP00960
```

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C          DERIVATIVES OF P(R) AND Q(R)  AT INTERIOR MESH POINTS.          PIP01450
CC                                                                        PIP01460
      H2ND = DMESH ** 2                                                   PIP01470
      ATEMP =  (-1.0D0/12.0D0) * H2ND                                     PIP01480
      A2DERQ(2)  = DCMPLX(ATEMP, 0.0D0)                                   PIP01490
      A2DERQ(6)  = A2DERQ(2)                                              PIP01500
      ATEMP =  (16.0D0/12.0D0) * H2ND                                     PIP01510
      A2DERQ(3)  = DCMPLX(ATEMP, 0.0D0)                                   PIP01520
      A2DERQ(5)  = A2DERQ(3)                                              PIP01530
      ATEMP = -2.5D0 * H2ND                                               PIP01540
      A2DERQ(4)  = DCMPLX(ATEMP, 0.0D0)                                   PIP01550
      A2DERP(1)  = A2DERQ(2)                                              PIP01560
      A2DERP(2)  = A2DERQ(3)                                              PIP01570
      A2DERP(3)  = A2DERQ(4)                                              PIP01580
      A2DERP(4)  = A2DERQ(5)                                              PIP01590
      A2DERP(5)  = A2DERQ(5)                                              PIP01600
C                                                                         PIP01610
CCC        COMPUTE CENTRAL DIFFERENCE COEFFICIENTS FOR FIRST ORDER        PIP01620
CCC                                                                       PIP01630
CCC        DERIVATIVES OF P(R) AND Q(R)  AT INTERIOR MESH POINTS.         PIP01640
C                                                                         PIP01650
      H1ST = DMESH                                                        PIP01660
      ATEMP =  (1.0D0/12.0D0) * H1ST                                      PIP01670
      A1DERQ(2)  = DCMPLX(ATEMP, 0.0D0)                                   PIP01680
      A1DERQ(6)  = -A1DERQ(2)                                             PIP01690
      ATEMP =  (-8.0D0/12.0D0) * H1ST                                     PIP01700
      A1DERQ(3)  = DCMPLX(ATEMP, 0.0D0)                                   PIP01710
      A1DERQ(5)  = -A1DERQ(3)                                             PIP01720
      A1DERP(1)  = A1DERQ(2)                                              PIP01730
      A1DERP(2)  = A1DERQ(3)                                              PIP01740
      A1DERP(4)  = A1DERQ(5)                                              PIP01750
      A1DERP(5)  = A1DERQ(5)                                              PIP01760
C                                                                         PIP01770
CC*************************************************************************PIP01780
CCC                                                                       PIP01790
CCC                          PART II                                      PIP01800
CCC                                                                       PIP01810
CCC        READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR   PIP01820
CCC                                                                       PIP01830
CCC        ALL DERIVATIVES OF P(R) AT POINTS NEAR THE AXIS.               PIP01840
CCC                                                                       PIP01850
CCC        BOUNDARY CONDITION AT THE AXIS IS: P(0) = 0.                   PIP01860
C                                                                         PIP01870
   41 READ(5,49) I,J,K                                                    PIP01880
      IF(I.EQ.9) GO TO 44                                                 PIP01890
      HTEMP = DMESH ** I                                                  PIP01900
      DO 42 KKK = 1,K                                                     PIP01910
      READ(5,*) AATEMP                                                    PIP01920
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      AATEMP = AATEMP * HTEMP                                          PIP01930
      AAXISP(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                           PIP01940
   42 CONTINUE                                                         PIP01950
      GO TO 41                                                         PIP01960
   44 CONTINUE                                                         PIP01970
C                                                                      PIP01980
CC       READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP01990
CCC                                                                    PIP02000
CCC      ALL DERIVATIVES OF P(R) AT POINTS NEAR THE WALL.              PIP02010
CCC                                                                    PIP02020
CCC      BOUNDARY CONDITION AT THE WALL IS:  P(1) = 0                  PIP02030
C                                                                      PIP02040
   45 READ(5,49) I,J,K                                                 PIP02050
      IF(I.EQ.9) GO TO 47                                              PIP02060
      HTEMP = DMESH ** I                                               PIP02070
      JJ = 3 - J                                                       PIP02080
      SIGN = 1.0D0                                                     PIP02090
      IF(I.EQ.1) SIGN = -SIGN                                          PIP02100
      DO 46 KKK = 1,K                                                  PIP02110
      KK = 6 - KKK                                                     PIP02120
      READ(5,*) WATEMP                                                 PIP02130
      WATEMP = WATEMP * HTEMP * SIGN                                   PIP02140
      AWALLP(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                           PIP02150
   46 CONTINUE                                                         PIP02160
      GO TO 45                                                         PIP02170
   47 CONTINUE                                                         PIP02180
   49 FORMAT(I1,1X,I1,1X,I1)                                           PIP02190
C                                                                      PIP02200
CC       READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP02210
CCC                                                                    PIP02220
CCC      ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE AXIS.              PIP02230
CCC                                                                    PIP02240
CCC      BOUNDARY CONDITIONS AT THE AXIS ARE: DQ(0) = 0 AND Q(0) = 0.  PIP02250
C                                                                      PIP02260
   51 READ(5,49) I,J,K                                                 PIP02270
      IF(I.EQ.9) GO TO 54                                              PIP02280
      HTEMP = DMESH ** I                                               PIP02290
      DO 52 KKK = 1,K                                                  PIP02300
      READ(5,*) AATEMP                                                 PIP02310
      AATEMP = AATEMP * HTEMP                                          PIP02320
      AAXISQ(I,J,KKK) = DCMPLX(AATEMP,0.0D0)                           PIP02330
   52 CONTINUE                                                         PIP02340
      GO TO 51                                                         PIP02350
   54 CONTINUE                                                         PIP02360
C                                                                      PIP02370
CC       READ IN PRE-COMPUTED NON-CENTRAL DIFFERENCE COEFFICIENTS FOR  PIP02380
CCC                                                                    PIP02390
C        ALL DERIVATIVES OF Q(R) AT POINTS NEAR THE WALL.              PIP02400
```

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C                                                                       PIP02410
C         BOUNDARY CONDITIONS AT THE WALL ARE: DQ(1) = 0 AND Q(1) = 0.  PIP02420
C                                                                       PIP02430
   55 READ(5,49) I,J,K                                                  PIP02440
      IF(I.EQ.9) GO TO 57                                               PIP02450
      HTEMP = DMESH ** I                                                PIP02460
      JJ = 4 - J                                                        PIP02470
      SIGN = 1.0D0                                                      PIP02480
      IF(I.EQ.1) SIGN = -SIGN                                           PIP02490
      IF(I.EQ.3) SIGN = -SIGN                                           PIP02500
      DO 56 KKK = 1,K                                                   PIP02510
      KK = 7 - KKK                                                      PIP02520
      READ(5,*) WATEMP                                                  PIP02530
      WATEMP = WATEMP * HTEMP * SIGN                                    PIP02540
      AWALLQ(I,JJ,KK) = DCMPLX(WATEMP,0.0D0)                            PIP02550
   56 CONTINUE                                                          PIP02560
      GO TO 55                                                          PIP02570
   57 CONTINUE                                                          PIP02580
C                                                                       PIP02590
CC        REARRANGE SOME OF THE DATA FOR DQ(R) AND D2Q(R) IN THE        PIP02600
CC                                                                      PIP02610
CC        AAXISQ AND AWALLQ COEFFICIENT MATRICES.                       PIP02620
C                                                                       PIP02630
      DO 58 I = 1,6                                                     PIP02640
      AAXISQ(2,3,I) = A2DERQ(I + 1)                                     PIP02650
      AAXISQ(1,3,I) = A1DERQ(I + 1)                                     PIP02660
      AWALLQ(2,1,I) = A2DERQ(I)                                         PIP02670
      AWALLQ(1,1,I) = A1DERQ(I)                                         PIP02680
   58 CONTINUE                                                          PIP02690
C                                                                       PIP02700
CC        READ IN ALPHA AND REYNOLDS NR                                 PIP02710
C                                                                       PIP02720
      IPLOT = 0                                                         PIP02730
   19 CONTINUE                                                          PIP02740
      READ(5,*) ALPHA,RE                                                PIP02750
      IF(ALPHA.LT.0.0D0) GO TO 180                                      PIP02760
C                                                                       PIP02770
C*********************************************************************  PIP02780
C                                                                       PIP02790
C                              PART III                                 PIP02800
C                                                                       PIP02810
C         COMPUTATION OF THE VORTICITY TRANSPORT EQUATION COEFFICIENTS  PIP02820
C                                                                       PIP02830
C         FOR P(R) AND Q(R) AND THEIR RESPECTIVE DERIVATIVES.           PIP02840
C                                                                       PIP02850
C         PRE-COMPUTE CONSTANTS INVOLVING ALPHA IN THE EQUATIONS.       PIP02860
C                                                                       PIP02870
      AL2 = ALPHA * ALPHA                                               PIP02880
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      AL12 = 12.0D0 * ALPHA                                                  PIP02890
      AL14 = 14.0D0 * ALPHA                                                  PIP02900
      AL24 = 24.0D0 * ALPHA                                                  PIP02910
      ALSQ = ALPHA * ALPHA                                                   PIP02920
      ALSQ2 = ALSQ + ALSQ                                                    PIP02930
      AL4TH = ALSQ * ALSQ                                                    PIP02940
      ALSQ6 = 6.0D0 * ALSQ                                                   PIP02950
      ALSQ9 = 9.0D0 * ALSQ                                                   PIP02960
      ALSQ24 = 24.0D0 * ALSQ                                                 PIP02970
      ALSQ36 = 36.0D0 * ALSQ                                                 PIP02980
      ALSQ57 = 57.0D0 * ALSQ                                                 PIP02990
      AL3RD2 = 2.0D0 * ALSQ * ALPHA                                          PIP03000
C                                                                            PIP03010
CC       COMPUTE THE INCREMENTAL PIPE RADIUS AT INTERIOR MESH POINTS.        PIP03020
C                                                                            PIP03030
      DO 60 I = 1,MESH                                                       PIP03040
      R = (DFLOAT(I) - 0.5D0) / DMESH                                        PIP03050
      R2 = R * R                                                             PIP03060
      R3 = R2 * R                                                            PIP03070
      R4 = R3 * R                                                            PIP03080
      R5 = R4 * R                                                            PIP03090
      RAD(I) = DCMPLX(R,0.0D0)                                               PIP03100
C                                                                            PIP03110
CC       VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)        PIP03120
CCC                                                                          PIP03130
CC       FIRST EQUATION  FIRST QUADRANT                                      PIP03140
C                                                                            PIP03150
      TEMPR = (R2 * (36.0D0 + ALSQ * R2)) / RE                               PIP03160
      D2P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03170
      TEMPR = ((252.0D0 * R) + (ALSQ9 * R2)) / RE                            PIP03180
      D1P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03190
      TEMPR = -(972.0D0 + (ALSQ57 * R2) + (AL4TH * R4)) / RE                 PIP03200
      TEMPI = - AL2 * R2 * (36.0D0 - (36.0D0 * R2) + (ALSQ * (R2 - R4)))     PIP03210
      D0P1(I) = DCMPLX(TEMPR,TEMPI)                                          PIP03220
      TEMPR = (36.0D0 * R2) + (ALSQ * R4)                                    PIP03230
      E0P1(I) = DCMPLX(TEMPR,0.0D0)                                          PIP03240
C                                                                            PIP03250
CC       V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R) AND D3Q(R)               PIP03260
CCC                                                                          PIP03270
CC       FIRST EQUATION  SECOND QUADRANT                                     PIP03280
C                                                                            PIP03290
      TEMPI = (6.0D0 * R2) / RE                                              PIP03300
      D3Q1(I) = DCMPLX(0.0D0,TEMPI)                                          PIP03310
      TEMPI = (66.0D0 * R) / RE                                              PIP03320
      D2Q1(I) = DCMPLX(0.0D0,TEMPI)                                          PIP03330
      TEMPR = AL12 *(R2 - R4)                                                PIP03340
      TEMPI = - (42.0D0 + (ALSQ6 * R2)) / RE                                 PIP03350
      D1Q1(I) = DCMPLX(TEMPR,TEMPI)                                          PIP03360
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      TEMPR = AL24 * ((2.0D0 * R) - R3)                                  PIP03370
      TEMPI = - ((768.0D0 / R) + (ALSQ36 * R)) / RE                      PIP03380
      D0Q1(I) = DCMPLX(TEMPR,TEMPI)                                      PIP03390
      TEMPI = 6.0D0 * R2                                                 PIP03400
      E1Q1(I) = DCMPLX(0.0D0,TEMPI)                                      PIP03410
      TEMPI = 24.0D0 * R                                                 PIP03420
      E0Q1(I) = DCMPLX(0.0D0,TEMPI)                                      PIP03430
C                                                                        PIP03440
CCCCC    VORTICITY TRANSPORT EQN COEFFICIENTS FOR P(R),DP(R) & D2P(R)    PIP03450
C                                                                        PIP03460
CCCCC    SECOND EQUATION  THIRD QUADRANT                                 PIP03470
C                                                                        PIP03480
      TEMPI = - (24.0D0 * R2) / RE                                       PIP03490
      D2P2(I) = DCMPLX(0.0D0,TEMPI)                                      PIP03500
      TEMPI = - (168.0D0 * R) / RE                                       PIP03510
      D1P2(I) = DCMPLX(0.0D0,TEMPI)                                      PIP03520
      TEMPR = - AL24 * (R2 - R4)                                         PIP03530
      TEMPI = (648.0D0 + (ALSQ24 * R2)) / RE                             PIP03540
      D0P2(I) = DCMPLX(TEMPR,TEMPI)                                      PIP03550
      TEMPI = - 12.0D0 * R2                                              PIP03560
      E0P2(I) = DCMPLX(0.0D0,TEMPI)                                      PIP03570
C                                                                        PIP03580
CCCCC    V T E COEFFICIENTS FOR Q(R), DQ(R), D2Q(R), D3Q(R) AND D4Q(R)   PIP03590
C                                                                        PIP03600
CCCCC    SECOND EQUATION  FOURTH QUADRANT                                PIP03610
C                                                                        PIP03620
      TEMPR = - R3 / RE                                                  PIP03630
      D4Q2(I) = DCMPLX(TEMPR,0.0D0)                                      PIP03640
      TEMPR = -(14.0D0 * R2) / RE                                        PIP03650
      D3Q2(I) = DCMPLX(TEMPR,0.0D0)                                      PIP03660
      TEMPR = (R * (21.0D0 + (ALSQ2 * R2)))/ RE                          PIP03670
      TEMPI = AL2 * (R3 - R5)                                            PIP03680
      D2Q2(I) = DCMPLX(TEMPR,TEMPI)                                      PIP03690
      TEMPR = (315.0D0 + (ALSQ14 * R2)) / RE                             PIP03700
      TEMPI = AL14 * (R2 - R4)                                           PIP03710
      D1Q2(I) = DCMPLX(TEMPR,TEMPI)                                      PIP03720
      TEMPR = - ((1152.0D0/R) + (ALS256 * R) + (AL4TH * R3)) / RE        PIP03730
      TEMPI = -((56.0D0 * ALPHA * (R - R3))+(AL3RD2 * (R3 - R5)))        PIP03740
      D0Q2(I) = DCMPLX(TEMPR,TEMPI)                                      PIP03750
      TEMPR = - R3                                                       PIP03760
      E2Q2(I) = DCMPLX(TEMPR,0.0D0)                                      PIP03770
      TEMPR = -7.0D0 * R2                                                PIP03780
      E1Q2(I) = DCMPLX(TEMPR,0.0D0)                                      PIP03790
      TEMPR = (28.0D0 * R) + (ALSQ * R3)                                 PIP03800
      E0Q2(I) = DCMPLX(TEMPR,0.0D0)                                      PIP03810
   60 CONTINUE                                                           PIP03820
C                                                                        PIP03830
C************************************************************************PIP03840
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C                                                                       PIP03850
CC                              PART IV                                 PIP03860
CC                                                                      PIP03870
CC        COMPUTATION OF THE A AND B MATRIX ELEMENTS.                   PIP03880
CC                                                                      PIP03890
CC        INITIALIZE THE A AND B MATRICES                               PIP03900
C                                                                       PIP03910
      DO 198 I = 1,100                                                  PIP03920
      DO 198 J = 1,100                                                  PIP03930
      A(I,J) = ZERO                                                     PIP03940
      B(I,J) = ZERO                                                     PIP03950
  198 CONTINUE                                                          PIP03960
C                                                                       PIP03970
CC        A AND B MATRIX ELEMENTS IN TERMS OF P(R)                      PIP03980
CC                                                                      PIP03990
CC        FIRST EQUATION  FIRST QUADRANT                                PIP04000
C                                                                       PIP04010
      DO 200 J = 1,2                                                    PIP04020
      DO 200 K = 1,5                                                    PIP04030
      AODER = ZERO                                                      PIP04040
      IF (K.EQ.J) AODER = ONE                                           PIP04050
      A(J,K) = AAXISP(2,J,K)*D2P1(J) + AAXISP(1,J,K)*D1P1(J) +          PIP04060
     1         AODER*D0P1(J)                                            PIP04070
      B(J,K) = AODER*E0P1(J)                                            PIP04080
  200 CONTINUE                                                          PIP04090
      MP = MESH - 2                                                     PIP04100
      DO 210 J = 3,MP                                                   PIP04110
      DO 210 K = 1,5                                                    PIP04120
      L = J - 3 + K                                                     PIP04130
      AODER = ZERO                                                      PIP04140
      IF(L.EQ.J) AODER = ONE                                            PIP04150
      A(J,L) = A2DERP(K)*D2P1(J) + A1DERP(K)*D1P1(J) + AODER*D0P1(J)    PIP04160
      B(J,L) = AODER*E0P1(J)                                            PIP04170
  210 CONTINUE                                                          PIP04180
      MMP = MP - 3                                                      PIP04190
      DO 220 J = 1,2                                                    PIP04200
      L = J + MP                                                        PIP04210
      DO 220 K = 1,5                                                    PIP04220
      N = MMP + K                                                       PIP04230
      AODER = ZERO                                                      PIP04240
      IF(N.EQ.L) AODER = ONE                                            PIP04250
      A(L,N) = AWALLP(2,J,K)*D2P1(L) + AWALLP(1,J,K)*D1P1(L) +          PIP04260
     1         AODER*D0P1(L)                                            PIP04270
      B(L,N) = AODER*E0P1(L)                                            PIP04280
  220 CONTINUE                                                          PIP04290
C                                                                       PIP04300
CC        A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                      PIP04310
C                                                                       PIP04320
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C          FIRST EQUATION  SECOND QUADRANT                                  PIP04330
C                                                                           PIP04340
      DO 250 J = 1,3                                                        PIP04350
      DO 250 K = 1,6                                                        PIP04360
      KK = K + MESH                                                         PIP04370
      AODER = ZERO                                                          PIP04380
      IF(K.EQ.J) AODER = ONE                                                PIP04390
      A(J,KK) =  AAXISQ(3,J,K)*D3Q1(J) + AAXISQ(2,J,K)*D2Q1(J) +            PIP04400
     1           AAXISQ(1,J,K)*D1Q1(J) + AODER*D0Q1(J)                      PIP04410
      B(J,KK) =  AAXISQ(1,J,K)*E1Q1(J) + AODER*E0Q1(J)                      PIP04420
  250 CONTINUE                                                              PIP04430
      M = MESH - 3                                                          PIP04440
      DO 260 J = 4,M                                                        PIP04450
      DO 260 K = 1,7                                                        PIP04460
      L = J - 4 + K                                                         PIP04470
      LL = L + MESH                                                         PIP04480
      AODER = ZERO                                                          PIP04490
      IF(L.EQ.J) AODER = ONE                                                PIP04500
      A(J,LL) =  A3DERQ(K)*D3Q1(J) + A2DERQ(K)*D2Q1(J) +                    PIP04510
     1           A1DERQ(K)*D1Q1(J) + AODER*D0Q1(J)                          PIP04520
      B(J,LL) =  A1DERQ(K)*E1Q1(J) + AODER*E0Q1(J)                          PIP04530
  260 CONTINUE                                                              PIP04540
      MM = M - 3                                                            PIP04550
      DO 265 J = 1,3                                                        PIP04560
      L = J + M                                                             PIP04570
      DO 265 K = 1,6                                                        PIP04580
      KK = 7 - K                                                            PIP04590
      N = MM + K                                                            PIP04600
      NN = N + MESH                                                         PIP04610
      AODER = ZERO                                                          PIP04620
      IF(N.EQ.L) AODER = ONE                                                PIP04630
      A(L,NN) =  AWALLQ(3,J,K)*D3Q1(L) + AWALLQ(2,J,K)*D2Q1(L) +            PIP04640
     1           AWALLQ(1,J,K)*D1Q1(L) + AODER*D0Q1(L)                      PIP04650
      B(L,NN) =  AWALLQ(1,J,K)*E1Q1(L) + AODER*E0Q1(L)                      PIP04660
  265 CONTINUE                                                              PIP04670
C                                                                           PIP04680
CCC       A AND B MATRIX ELEMENTS IN TERMS OF P(R)                          PIP04690
CCC                                                                         PIP04700
CCC       SECOND EQUATION  THIRD QUADRANT                                   PIP04710
C                                                                           PIP04720
      DO 270 J = 1,2                                                        PIP04730
      JJ = J + MESH                                                         PIP04740
      DO 270 K = 1,5                                                        PIP04750
      AODER = ZERO                                                          PIP04760
      IF(K.EQ.J) AODER = ONE                                                PIP04770
      A(JJ,K) =  AAXISP(2,J,K)*D2P2(J) + AAXISP(1,J,K)*D1P2(J) +            PIP04780
     1           AODER*D0P2(J)                                              PIP04790
      B(JJ,K) =  AODER*E0P2(J)                                              PIP04800
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  270 CONTINUE                                                                  PIP04810
      DO 275 J = 3,MP                                                           PIP04820
      JJ = J + MESH                                                             PIP04830
      DO 275 K = 1,5                                                            PIP04840
      L = J - 3 + K                                                             PIP04850
      AODER = ZERO                                                              PIP04860
      IF(L.EQ.J) AODER = ONE                                                    PIP04870
      A(JJ,L) = A2DERP(K)*D2P2(J) + A1DERP(K)*D1P2(J) + AODER*DOP2(J)           PIP04880
      B(JJ,L) = AODER*EOP2(J)                                                   PIP04890
  275 CONTINUE                                                                  PIP04900
      DO 280 J = 1,2                                                            PIP04910
      L = J + MP                                                                PIP04920
      LL = L + MESH                                                             PIP04930
      DO 280 K = 1,5                                                            PIP04940
      N = MMP + K                                                               PIP04950
      AODER = ZERO                                                              PIP04960
      IF (N.EQ.L) AODER = ONE                                                   PIP04970
      A(LL,N) = AWALLP(2,J,K)*D2P2(L) + AWALLP(1,J,K)*D1P2(L) +                 PIP04980
     1          AODER*DOP2(L)                                                   PIP04990
      B(LL,N) = AODER*EOP2(L)                                                   PIP05000
  280 CONTINUE                                                                  PIP05010
C                                                                               PIP05020
CCC       A AND B MATRIX ELEMENTS IN TERMS OF Q(R)                              PIP05030
CCC                                                                             PIP05040
CCC       SECOND EQUATION  FOURTH QUADRANT                                      PIP05050
C                                                                               PIP05060
      DO 285 J = 1,3                                                            PIP05070
      JJ = J + MESH                                                             PIP05080
      DO 285 K = 1,6                                                            PIP05090
      KK = K + MESH                                                             PIP05100
      AODER = ZERO                                                              PIP05110
      IF(K.EQ.J) AODER = ONE                                                    PIP05120
      A(JJ,KK) = AAXISQ(4,J,K)*D4Q2(J) + AAXISQ(3,J,K)*D3Q2(J) +                PIP05130
     1           AAXISQ(2,J,K)*D2Q2(J) + AAXISQ(1,J,K)*D1Q2(J) +                PIP05140
     2           AODER*DOQ2(J)                                                  PIP05150
      B(JJ,KK) = AAXISQ(2,J,K)*E2Q2(J) + AAXISQ(1,J,K)*E1Q2(J) +                PIP05160
     1           AODER*EOQ2(J)                                                  PIP05170
  285 CONTINUE                                                                  PIP05180
      DO 290 J = 4,M                                                            PIP05190
      JJ = J + MESH                                                             PIP05200
      DO 290 K = 1,7                                                            PIP05210
      L = J - 4 + K                                                             PIP05220
      LL = L + MESH                                                             PIP05230
      AODER = ZERO                                                              PIP05240
      IF(L.EQ.J) AODER = ONE                                                    PIP05250
      A(JJ,LL) = A4DERQ(K)*D4Q2(J) + A3DERQ(K)*D3Q2(J) +                        PIP05260
     1           A2DERQ(K)*D2Q2(J) + A1DERQ(K)*D1Q2(J) + AODER*DOQ2(J)          PIP05270
      B(JJ,LL) = A2DERQ(K)*E2Q2(J) + A1DERQ(K)*E1Q2(J) + AODER*EOQ2(J)          PIP05280
```

```
  290 CONTINUE                                                          PIP05290
      DO 295 J = 1,3                                                    PIP05300
      L = J + M                                                         PIP05310
      LL = L + MESH                                                     PIP05320
      DO 295 K = 1,6                                                    PIP05330
      KK = 7 - K                                                        PIP05340
      N = MM + K                                                        PIP05350
      NN = N + MESH                                                     PIP05360
      AODER = ZERO                                                      PIP05370
      IF(N.EQ.L)  AODER = ONE                                           PIP05380
      A(LL,NN) = AWALLQ(4,J,K)*D4Q2(L) + AWALLQ(3,J,K)*D3Q2(L) +        PIP05390
     1           AWALLQ(2,J,K)*D2Q2(L) + AWALLQ(1,J,K)*D1Q2(L) +        PIP05400
     2           AODER*DOQ2(L)                                          PIP05410
      B(LL,NN) = AWALLQ(2,J,K)*E2Q2(L) + AWALLQ(1,J,K)*E1Q2(L) +        PIP05420
     1           AODER*EOQ2(L)                                          PIP05430
  295 CONTINUE                                                          PIP05440
C                                                                       PIP05450
CC*********************************************************************PIP05460
C                                                                       PIP05470
C                              PART V                                   PIP05480
C                                                                       PIP05490
C        COMPUTATION OF EIGENVALUES AND RESPECTIVE EIGENVECTORS         PIP05500
C                                                                       PIP05510
C        USING THE SUBROUTINE EIGZC WITH THE VORTICITY TRANSPORT        PIP05520
C                                                                       PIP05530
C        EQUATIONS IN THE FORM  A * X = GAMMA * B * X.                  PIP05540
C                                                                       PIP05550
      DO 140 I = 1,100                                                  PIP05560
      EIGVAL(I) = ZERO                                                  PIP05570
      EIGA(I) = ZERO                                                    PIP05580
      EIGB(I) = ZERO                                                    PIP05590
      DO 140 J = 1,100                                                  PIP05600
      EIGVEC(I,J) = ZERO                                                PIP05610
      AA(I,J) = A(I,J)                                                  PIP05620
      BB(I,J) = B(I,J)                                                  PIP05630
  140 CONTINUE                                                          PIP05640
      WRITE(6,145)                                                      PIP05650
  145 FORMAT(1X,//,1X,'*******************************',                PIP05660
     1   '*********************************************',              PIP05670
     2   '*********************************************',//)           PIP05680
      MESH2 = (MESH * 2)                                                PIP05690
      IJOB = 1                                                          PIP05700
      CALL EIGZC(AA,100,BB,100,MESH2,IJOB,EIGA,EIGB,                    PIP05710
     1EIGVEC,100,WK,INEER,IER)                                          PIP05720
      IF(INEER.EQ.0)  WRITE(6,150)                                      PIP05730
  150 FORMAT(1X,////,10X,'ALL EIGENVALUES CONVERGED WITHIN 30 ',        PIP05740
     1       'ITERATIONS',//)                                           PIP05750
      IF(INEER.NE.0)  WRITE(6,155) INEER                                PIP05760
```

```
  155 FORMAT(1X,////,10X,'FAILED TO CONVERGE ON THE',I4,1X,                PIP05770
     1       'TH EIGENVALUE',//)                                           PIP05780
      DO 165 I = 1,MESH2                                                   PIP05790
      EIGVAL(I) = EIGA(I)/EIGB(I)                                          PIP05800
      GAMR(I) = DREAL(EIGVAL(I))                                           PIP05810
  165 CONTINUE                                                             PIP05820
      NGAM = 1                                                             PIP05830
      GAMMAX = GAMR(1)                                                     PIP05840
      DO 170 I = 2,MESH2                                                   PIP05850
      IF(GAMR(I).LT.GAMMAX) GO TO 169                                      PIP05860
      GAMMAX = GAMR(I)                                                     PIP05870
      NGAM = I                                                             PIP05880
  169 CONTINUE                                                             PIP05890
  170 CONTINUE                                                             PIP05900
      WRITE(6,175) NGAM,GAMMAX,MESH,ALPHA,RE                               PIP05910
  175 FORMAT(1X,/,1X,'LEAST STABLE EIGENVALUE NR',I3,1X,'=',D27.18,4X,     PIP05920
     1 'MESH =',I3,4X,'ALPHA =',D10.3,4X,'REYNOLDS NR =',D17.10,/)         PIP05930
C                                                                          PIP05940
C*************************************************************************PIP05950
C                                                                          PIP05960
C                              PART VI                                     PIP05970
C                                                                          PIP05980
C        VERIFY THAT THE LEAST STABLE EIGENVALUE AND CORRESPONDING         PIP05990
C                                                                          PIP06000
C        EIGENVECTOR SATISFY THE GOVERNING PARTIAL DIFFERENTIAL            PIP06010
C                                                                          PIP06020
C        EQUATIONS.  RESIDUAL ERROR IS OUTPUT AS EPSILON.                  PIP06030
C                                                                          PIP06040
      DO 351 I = 1,MESH2                                                   PIP06050
      SUMA = ZERO                                                          PIP06060
      SUMB = ZERO                                                          PIP06070
      DO 350 J = 1,MESH2                                                   PIP06080
      SUMA = SUMA + ( A(I,J) * EIGVEC(J,NGAM) )                            PIP06090
      SUMB = SUMB + ( B(I,J) * EIGVEC(J,NGAM) )                            PIP06100
  350 CONTINUE                                                             PIP06110
      ERROR(I) = SUMA - ( SUMB * EIGVAL(NGAM) )                            PIP06120
  351 CONTINUE                                                             PIP06130
      WRITE(6,375)                                                         PIP06140
  375 FORMAT(1X,//,15X,'EPSILON = A * X  -  GAMMA * B * X',/)              PIP06150
      WRITE(6,380) (ERROR(I),I = 1,MESH2)                                  PIP06160
  380 FORMAT(1X,2D30.18)                                                   PIP06170
C                                                                          PIP06180
C*************************************************************************PIP06190
C                                                                          PIP06200
C                              PART VII                                    PIP06210
C                                                                          PIP06220
C        COMPUTATION OF AXIAL PERTURBATION VELOCITY  U(R).                 PIP06230
C                                                                          PIP06240
```

```
C          OUTPUT THE EIGENVECTOR CORRESPONDING TO THE LEASE STABLE      PIP06250
CC                                                                       PIP06260
CCC        EIGENVALUE,  GAMMA*.                                          PIP06270
C                                                                        PIP06280
      DO 400 I = 1,MESH                                                  PIP06290
      II = I + MESH                                                      PIP06300
      P(I) = EIGVEC(I,NGAM)                                              PIP06310
      Q(I) = EIGVEC(II,NGAM)                                             PIP06320
  400 CONTINUE                                                           PIP06330
      WRITE(6,401) EIGVAL(NGAM)                                          PIP06340
  401 FORMAT(1X,///,5X,'EIGENVECTOR CORRESPONDING TO',                   PIP06350
     1       ' THE LEAST STABLE EIGENVALUE, GAMMA* =',2D27.18,/)         PIP06360
      DO 402 I = 1,MESH                                                  PIP06370
      II = I + MESH                                                      PIP06380
      WRITE(6,404) I,P(I),II,Q(I)                                        PIP06390
  404 FORMAT(2X,I2,2D28.18,2X,I2,2D28.18)                                PIP06400
  402 CONTINUE                                                           PIP06410
C                                                                        PIP06420
CCC        COMPUTATION OF DQ(R)                                          PIP06430
C                                                                        PIP06440
      DO 410 J = 1,3                                                     PIP06450
      DQ(J) = ZERO                                                       PIP06460
      DO 410 K = 1,6                                                     PIP06470
      DQ(J) = DQ(J) +(Q(K) * AAXISQ(1,J,K))                              PIP06480
  410 CONTINUE                                                           PIP06490
C                                                                        PIP06500
      DO 420 J = 4,M                                                     PIP06510
      DQ(J) = ZERO                                                       PIP06520
      DO 420 K = 1,7                                                     PIP06530
      L = J - 4 + K                                                      PIP06540
      DQ(J) = DQ(J) +(Q(L) * A1DERQ(K))                                  PIP06550
  420 CONTINUE                                                           PIP06560
C                                                                        PIP06570
      DO 430 J = 1,3                                                     PIP06580
      L = J + M                                                          PIP06590
      DQ(L) = ZERO                                                       PIP06600
      DO 430 K = 1,6                                                     PIP06610
      KK = 7 - K                                                         PIP06620
      N = MM + K                                                         PIP06630
      DQ(L) = DQ(L) +(Q(N) * AWALLQ(1,J,K))                              PIP06640
  430 CONTINUE                                                           PIP06650
C                                                                        PIP06660
      DO 440 I = 1,MESH                                                  PIP06670
      U(I) = (RAD(I) ** 2) * ((FOUR * Q(I)) + (RAD(I) * DQ(I)) -         PIP06680
     1       (SIXIMG * P(I) * RAD(I)))                                   PIP06690
  440 CONTINUE                                                           PIP06700
C                                                                        PIP06710
C          REARRANGE DATA FOR PLOT ROUTINE                               PIP06720
```

```
C                                                                       PIP06730
      M2 = MESH + 2                                                     PIP06740
      DO 450 I = 1,MESH                                                 PIP06750
      II = M2 - I                                                       PIP06760
      III = II - 1                                                      PIP06770
      U(II) = U(III)                                                    PIP06780
      RAD(II) = RAD(III)                                                PIP06790
  450 CONTINUE                                                          PIP06800
      RAD(1) = ZERO                                                     PIP06810
      RAD(M2) = ONE                                                     PIP06820
      U(1) = ZERO                                                       PIP06830
      U(M2) = ZERO                                                      PIP06840
      DO 460 I = 1,M2                                                   PIP06850
      RADR(I) = DREAL(RAD(I))                                           PIP06860
  460 CONTINUE                                                          PIP06870
C                                                                       PIP06880
CC       COMPUTE THE NORMALIZED PERTURBATION VELOCITY AND               PIP06890
CC                                                                      PIP06900
CC       NORMALIZED AMPLITUDE AND OUTPUT THE RESULTS.                   PIP06910
C                                                                       PIP06920
      DO 465 I = 1,M2                                                   PIP06930
      UAMP(I) = CDABS(U(I))                                             PIP06940
  465 CONTINUE                                                          PIP06950
      AMPMAX = UAMP(1)                                                  PIP06960
      IMAX = 1                                                          PIP06970
      DO 470 J = 2,M2                                                   PIP06980
      AMTEMP = UAMP(J)                                                  PIP06990
      IF(AMTEMP.LT.AMPMAX) GO TO 471                                    PIP07000
      AMPMAX = UAMP(J)                                                  PIP07010
      IMAX = J                                                          PIP07020
  471 CONTINUE                                                          PIP07030
  470 CONTINUE                                                          PIP07040
      WRITE(6,472)                                                      PIP07050
  472 FORMAT(1X,///,23X,'NORMALIZED PERTURBATION VELOCITY',             PIP07060
     1 24X,'NORMALIZED AMPLITUDE',/)                                    PIP07070
      DO 475 K = 1,M2                                                   PIP07080
      UAMNOR(K) = UAMP(K) / AMPMAX                                      PIP07090
      AMPNOR(K) = UAMNOR(K)                                             PIP07100
      UNORM(K) = U(K) / U(IMAX)                                         PIP07110
      URDP(K) = DREAL(UNORM(K))                                         PIP07120
      UR(K) = URDP(K)                                                   PIP07130
      UIDP(K) = DIMAG(UNORM(K))                                         PIP07140
      UI(K) = UIDP(K)                                                   PIP07150
      KK = K - 1                                                        PIP07160
      WRITE(6,466) KK,URDP(K),UIDP(K),UAMNOR(K)                         PIP07170
  466 FORMAT(1X,I3,1X,2D30.18,5X,D30.18)                                PIP07180
  475 CONTINUE                                                          PIP07190
C                                                                       PIP07200
```

```
C*******************************************************************************  PIP07210
CCC                                                                               PIP07220
CCC                              PART VIII                                        PIP07230
CCCCC                                                                             PIP07240
CCC         PLOT THE NORMALIZED PERTURBATION VELOCITY VS PIPE RADIUS.             PIP07250
C                                                                                 PIP07260
      IPLOT = IPLOT + 1                                                           PIP07270
      RMESH = FLOAT(MESH)                                                         PIP07280
      ALPLOT = ALPHA                                                              PIP07290
      REPLOT = RE                                                                 PIP07300
      GAMPLT = GAMMAX                                                             PIP07310
      CALL PLOTG (RADR,UR,M2,1,1,1,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)          PIP07320
      CALL PLOTG (RADR,UI,M2,2,1,5,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)          PIP07330
C     CALL PLOTG (RADR,AMPNDR,M2,3,1,2,' ',1,' ',1,0.,1.,-1.,1.,5.,5.)            PIP07340
      CALL PLOTG (X1,Y1,2,3,1,0,' ',1,' ',1,0.0,1.0,-1.0,1.0,5.0,5.0)             PIP07350
      XO = 0.15                                                                   PIP07360
      YO = 5.6                                                                    PIP07370
      IF(IPLOT.EQ.4)  IPLOT = 1                                                   PIP07380
      IF(IPLOT.EQ.2)  YO = 12.1                                                   PIP07390
      IF(IPLOT.EQ.3)  YO = 18.6                                                   PIP07400
      CALL TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                               PIP07410
      GO TO 19                                                                    PIP07420
  180 CONTINUE                                                                    PIP07430
      CALL PLOT(0.0,0.0,999)                                                      PIP07440
      STOP                                                                        PIP07450
      END                                                                         PIP07460
C                                                                                 PIP07470
CC.................................................................               PIP07480
CCCCC                                                                             PIP07490
CCC         SUBROUTINE TITLE1                                                     PIP07500
CCCCC                                                                             PIP07510
CCC         SUBROUTINE TO WRITE TITLE ON PLOTS                                    PIP07520
C                                                                                 PIP07530
      SUBROUTINE TITLE1(XO,YO,RMESH,ALPLOT,REPLOT,GAMPLT)                         PIP07540
      RN = 6.0                                                                    PIP07550
      YHT = 0.14                                                                  PIP07560
      CALL NEWPEN(2)                                                              PIP07570
      CALL SYMBOL(XO,YO,YHT,                                                      PIP07580
     1'NORMALIZED PERTURBATION VELOCITY VS RADIUS',0.0,42)                        PIP07590
      YHT = 0.08                                                                  PIP07600
      DELY = 0.20                                                                 PIP07610
      CALL NEWPEN(1)                                                              PIP07620
      YO = YO - 0.8                                                               PIP07630
      XO = XO + 2.5                                                               PIP07640
      XN = XO + .75                                                               PIP07650
      CALL SYMBOL(XO,YO,YHT,'N      = ',0.0,9)                                    PIP07660
      CALL NUMBER(XN,YO,YHT,RN,0.0,-1)                                            PIP07670
      YO = YO - DELY                                                              PIP07680
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      CALL SYMBOL(XO,YO,YHT,'MESH    = ',0.0,9)                                  PIP07690
      CALL NUMBER(XN,YO,YHT,RMESH,0.0,-1)                                        PIP07700
      YO = YO - DELY                                                             PIP07710
      CALL SYMBOL(XO,YO,YHT,'ALPHA   = ',0.0,9)                                  PIP07720
      CALL NUMBER(XN,YO,YHT,ALPLOT,0.0,2)                                        PIP07730
      YO = YO - DELY                                                             PIP07740
      CALL SYMBOL(XO,YO,YHT,'REY NR  = ',0.0,9)                                  PIP07750
      CALL NUMBER(XN,YO,YHT,REPLOT,0.0,1)                                        PIP07760
      YO = YO - DELY                                                             PIP07770
      CALL SYMBOL(XO,YO,YHT,'GAMMA*  = ',0.0,9)                                  PIP07780
      CALL NUMBER(XN,YO,YHT,GAMPLT,0.0,4)                                        PIP07790
      YO = YO - 3.4                                                              PIP07800
      CALL SYMBOL(XO,YO,YHT,'OCTAGON = U(REAL)',0.0,17)                          PIP07810
      YO = YO - DELY                                                             PIP07820
      CALL SYMBOL(XO,YO,YHT,'DIAMOND = U(IMAG)',0.0,17)                          PIP07830
      YO = YO - DELY                                                             PIP07840
C     CALL SYMBOL(XO,YO,YHT,'TRIANGLE= U AMPL ',0.0,17)                          PIP07850
C                                                                                PIP07860
CCC      WRITE THE AXIS LABLES                                                   PIP07870
C                                                                                PIP07880
      XO = XO - 2.55                                                             PIP07890
      YO = YO + 0.3                                                              PIP07900
      YHT = 0.12                                                                 PIP07910
      CALL SYMBOL(XO,YO,YHT,'NORMALIZED PERTURBATION VELOCITY',90.,32)           PIP07920
      XO = XO + 2.8                                                              PIP07930
      YO = YO - 1.0                                                              PIP07940
      CALL SYMBOL(XO,YO,YHT,'PIPE RADIUS',0.0,11)                                PIP07950
      RETURN                                                                     PIP07960
      END                                                                        PIP07970
/*                                                                               PIP07980
//GO.SYSIN DD *                                                                  PIP07990
2 1 5                                                                            PIP08000
     -0.54166666666666658D+01                                                    PIP08010
      0.24444444444444417D+01                                                    PIP08020
     -0.49999999999999948D+00                                                    PIP08030
      0.95238095238094885D-01                                                    PIP08040
     -0.92592592592588987D-02                                                    PIP08050
2 2 5                                                                            PIP08060
      0.17499999999999924D+01                                                    PIP08070
     -0.27777777777777590D+01                                                    PIP08080
      0.14999999999999917D+01                                                    PIP08090
     -0.14285714285713968D+00                                                    PIP08100
      0.92592592592587591D-02                                                    PIP08110
1 1 4                                                                            PIP08120
      0.16666666666666586D+00                                                    PIP08130
      0.10000000000000002D+01                                                    PIP08140
     -0.29999999999999989D+00                                                    PIP08150
      0.47619047619047633D-01                                                    PIP08160
```

```
1 2 4
      -0.99999999999997335D+00
       0.16666666666666429D+00
       0.60000000000001199D+00
      -0.71428571428571285D-01
9 0 0
2 1 5
      -0.54166666666665808D+01
       0.24444444444444176D+01
      -0.49999999999994837D+00
       0.95238095238094885D-01
      -0.92592592592588987D-02
2 2 5
       0.17499999999999245D+01
      -0.27777777777777590D+01
       0.14999999999999173D+01
      -0.14285714285713968D+00
       0.92592592592587591D-02
1 1 4
       0.16666666666666586D+00
       0.10000000000000022D+01
      -0.29999999999999892D+00
       0.47619047619047638D-01
1 2 4
      -0.99999999999997335D+00
       0.16666666666666429D+00
       0.60000000000001199D+00
      -0.71428571428571285D-01
9 0 0
4 1 6
       0.11500000000000071D+03
      -0.38444444444445025D+02
       0.16400000000000385D+02
      -0.52244897959184979D+01
       0.11234567901234950D+01
      -0.11570247933884756D+00
4 2 6
      -0.26000000000000802D+02
       0.17666666666667016D+02
      -0.11200000000000230D+02
       0.39591836734695107D+01
      -0.66666666666670049D+00
       0.57851239669425180D-01
4 3 6
       0.10000000000000852D+01
      -0.62222222222228040D+01
       0.92000000000004575D+01
      -0.64489795918368990D+01
```

```
      0.19876543209876977 9D+01
     -0.16528925619835255 2D+00
3 1 5
     -0.17499999999999975 1D+02
      0.11111111111110760 8D+00
      0.12000000000000137 2D+01
     -0.34693877551020739 5D+00
      0.43209876543210391 1D-01
3 2 5
      0.13499999999999968 0D+02
     -0.38888888888888821 8D+01
     -0.14210854715202003 7D-13
      0.55102040816326658 4D+00
     -0.67901234567901785 3D-01
3 3 5
     -0.15500000000000113 7D+02
      0.98888888888891060 D+01
     -0.60000000000000532 9D+01
      0.14489795918367498 5D+01
      0.67901234567899036 D-01
2 1 4
     -0.46666666666666585 0D+01
      0.21111111111111027 0D+01
     -0.31999999999999993 0D+00
      0.34013605442176852 1D-01
2 2 4
      0.99999999999978684 0D+00
     -0.24444444444444493 0D+01
      0.13200000000000117 0D+01
     -0.81632653061224511 0D-01
1 1 3
      0.24999999999999883 0D+01
      0.22222222222222293 0D+00
     -0.19999999999999978 0D-01
1 2 3
     -0.45000000000000355 0D+01
      0.13333333333333304 0D+01
      0.18000000000000007 0D+00
9 0 0
4 1 6
      0.11500000000000071 4D+03
     -0.38444444444445025 5D+02
      0.16400000000000385 3D+02
     -0.52244897959184979 4D+01
      0.11234567901234950 9D+01
     -0.11570247933884755 8D+00
4 2 6
     -0.26000000000000802 9D+02
```

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PIP08650
PIP08660
PIP08670
PIP08680
PIP08690
PIP08700
PIP08710
PIP08720
PIP08730
PIP08740
PIP08750
PIP08760
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PIP08980
PIP08990
PIP09000
PIP09010
PIP09020
PIP09030
PIP09040
PIP09050
PIP09060
PIP09070
PIP09080
PIP09090
PIP09100
PIP09110
PIP09120
```

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      0.176666666666670160D+02                                PIP09130
     -0.112000000000023020D+02                                PIP09140
      0.395918367346951072D+01                                PIP09150
     -0.666666666666700491D+00                                PIP09160
      0.578512396694251807D-01                                PIP09170
 4 3 6                                                        PIP09180
      0.100000000000085265D+01                                PIP09190
     -0.622222222222280408D+01                                PIP09200
      0.920000000000045759D+01                                PIP09210
     -0.644897959183689906D+01                                PIP09220
      0.198765432098769779D+01                                PIP09230
     -0.165289256198352552D+00                                PIP09240
 3 1 5                                                        PIP09250
     -0.174999999999997751D+02                                PIP09260
      0.111111111111107608D+00                                PIP09270
      0.120000000000001372D+01                                PIP09280
     -0.346938775510207395D+00                                PIP09290
      0.432098765432103911D-01                                PIP09300
 3 2 5                                                        PIP09310
      0.134999999999999680D+02                                PIP09320
     -0.388888888888888218D+01                                PIP09330
     -0.142108547152020037D-13                                PIP09340
      0.551020408163266584D+00                                PIP09350
     -0.679012345679017853D-01                                PIP09360
 3 3 5                                                        PIP09370
     -0.155000000000001137D+02                                PIP09380
      0.988888888888891060D+01                                PIP09390
     -0.600000000000005329D+01                                PIP09400
      0.144897959183674985D+01                                PIP09410
      0.679012345678990935D-01                                PIP09420
 2 1 4                                                        PIP09430
     -0.466666666666666585D+01                                PIP09440
      0.211111111111111027D+01                                PIP09450
     -0.319999999999999993D+00                                PIP09460
      0.340136054421768521D-01                                PIP09470
 2 2 4                                                        PIP09480
      0.999999999999978684D+00                                PIP09490
     -0.244444444444444930D+01                                PIP09500
      0.132000000000000117D+01                                PIP09510
     -0.816326530612245110D-01                                PIP09520
 1 1 3                                                        PIP09530
      0.249999999999999889D+01                                PIP09540
      0.222222222222222293D+00                                PIP09550
     -0.199999999999999978D-01                                PIP09560
 1 2 3                                                        PIP09570
     -0.450000000000000355D+01                                PIP09580
      0.133333333333333304D+01                                PIP09590
      0.180000000000000007D+00                                PIP09600
```

```
9 0 0                                                                   PIP09610
0.0D0       1150.0D0                                                    PIP09620
0.5D0       1150.0D0                                                    PIP09630
1.0D0       1150.0D0                                                    PIP09640
2.0D0       1150.0D0                                                    PIP09650
4.0D0       1150.0D0                                                    PIP09660
8.0D0       1150.0D0                                                    PIP09670
16.0D0      1150.0D0                                                    PIP09680
32.0D0      1150.0D0                                                    PIP09690
0.0D0       2300.0D0                                                    PIP09700
0.5D0       2300.0D0                                                    PIP09710
1.0D0       2300.0D0                                                    PIP09720
2.0D0       2300.0D0                                                    PIP09730
4.0D0       2300.0D0                                                    PIP09740
8.0D0       2300.0D0                                                    PIP09750
16.0D0      2300.0D0                                                    PIP09760
32.0D0      2300.0D0                                                    PIP09770
-1.0D0      0.0D0                                                       PIP09780
/*                                                                      PIP09790
//                                                                      PIP09800
```

```
C*************************************************************************  COE00010
C                                                                          COE00020
C       THIS IS A PROGRAM USED TO COMPUTE THE NONCENTRAL FINITE            COE00030
C       DIFFERENCE COEFFICIENTS FOR THE DERIVATIVES OF THE FUNCTIONS       COE00040
C       P(R) AND Q(R) FOR THE THREE CASES INVESTIGATED.  THE DATA          COE00050
C       IS THEN READ INTO THE MAIN INVESTIGATIVE PROGRAMS FOR N = 0,       COE00060
C       1 AND 6.  THE DATA IS ANNOTATED AS TO IT'S APPLICATION             COE00070
C       DEPENDING ON THE BOUNDARY CONDITIONS AT THE AXIS OR WALL.          COE00080
C       THESE ANNOTATIONS SHOULD BE REMOVED BEFORE THE DATA IS USED.       COE00090
C       THE PROGRAM IS RUN IN THE VM MODE AND THE DATA IS OUTPUT           COE00100
C       TO A FILE ON THE USER'S DISK WITH A FILENAME = COEFF AND           COE00110
C       FILETYPE = DATAOUT2.  THIS IS ACCOMPLISHED WITH THE USE OF         COE00120
C       SPECIAL EXEC FILE CALLED,  RUNDATA.  THIS EXEC FILE IS             COE00130
C       LISTED AT THE END OF THE COEFFICIENT PROGRAM.  THE PROPER          COE00140
C       COMMAND TO RUN THE PROGRAM IS,  RUNDATA COEFF T.                   COE00150
C*************************************************************************  COE00160
C                                                                          COE00170
      REAL*8 A6(6,6),FACT,H(6),XJ,XI,A6INV(6,6),WKAREA(60)                 COE00180
      REAL*8 A5(5,5),A4(4,4),A5INV(5,5),A4INV(4,4),A3(3,3),A3INV(3,3)      COE00190
      REAL*8 C(3,4,6)                                                      COE00200
      REAL*8 C4X61(4,6),C4X62(4,6),C4X63(4,6)                              COE00210
      REAL*8 C4X51(4,5),C4X52(4,5),C4X53(4,5)                              COE00220
      REAL*8 C3X51(3,5),C3X52(3,5),C3X53(3,5)                              COE00230
      REAL*8 C4X41(4,4),C4X42(4,4)                                         COE00240
      REAL*8 C2X41(2,4),C2X42(2,4)                                         COE00250
      REAL*8 C3X31(3,3),C3X32(3,3)                                         COE00260
      REAL*8 C1X31(1,3),C1X32(1,3)                                         COE00270
      REAL*8 C46A61(4,6),C46A62(4,6),C46A63(4,6)                           COE00280
      REAL*8 C45A51(4,5),C45A52(4,5),C45A53(4,5)                           COE00290
      REAL*8 C35A51(3,5),C35A52(3,5),C35A53(3,5)                           COE00300
      REAL*8 C44A41(4,4),C44A42(4,4)                                       COE00310
      REAL*8 C24A41(2,4),C24A42(2,4)                                       COE00320
      REAL*8 C33A31(3,3),C33A32(3,3)                                       COE00330
      REAL*8 C13A31(1,3),C13A32(1,3)                                       COE00340
      REAL*8 C2X51(2,5),C2X52(2,5)                                         COE00350
```

```
      REAL*8 C1X41(1,4),C1X42(1,4)                                      CDBE00490
      REAL*8 C25A51(2,5),C25A52(2,5)                                    CDBE00500
      REAL*8 C14A41(1,4),C14A42(1,4)                                    CDBE00510
C                                                                       CDBE00520
CCC      THE AA MATRICES ARE 6 X 6, 5 X 5, 4 X 4 & 3 X 3  AND ARE       CDBE00530
CCC                                                                     CDBE00540
CCC      CALLED A6,A5,A4 & A3, RESPECTIVELY.                            CDBE00550
C                                                                       CDBE00560
      DO 10 I=1,6                                                       CDBE00570
      FACT = 1.0D0                                                      CDBE00580
      XI = DFLOAT(I)                                                    CDBE00590
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                     CDBE00600
      DO 10 J=1,6                                                       CDBE00610
      XJ = DFLOAT(J)                                                    CDBE00620
      FACT = FACT*(XJ+1.0D0)                                            CDBE00630
      A6(I,J) = (H(I)**(J+1))/FACT                                      CDBE00640
   10 CONTINUE                                                          CDBE00650
   20 FORMAT(1X,D29.18)                                                 CDBE00660
C                                                                       CDBE00670
CC*********************************************************************CDBE00680
CCC                                                                     CDBE00690
CCC      A SUBPROGRAM USED TO COMPUTE THE NONCENTRAL DIFFERENCE         CDBE00700
CCC                                                                     CDBE00710
CCC      COEFFICIENTS THE AXIS FOR N = 0.                               CDBE00720
CCC                                                                     CDBE00730
CCC      THE APPLICABLE BOUNDARY CONDITIONS ARE:                        CDBE00740
CCC                                                                     CDBE00750
CCC      DQ(0) = 0 AND D3Q(0) = 0  FOR N = 0 AT THE AXIS.               CDBE00760
CCC                                                                     CDBE00770
CC*********************************************************************CDBE00780
CCC                                                                     CDBE00790
CCC      COMPUTE THE AA MATRICES AND THEIR INVERSES.                    CDBE00800
C                                                                       CDBE00810
      DO 11 I = 1,6                                                     CDBE00820
      A6(I,2) = A6(I,1)                                                 CDBE00830
      A6(I,1) = 1.0D0                                                   CDBE00840
   11 CONTINUE                                                          CDBE00850
      CALL LINV2F(A6,6,6,A6INV,10,WKAREA,IER)                           CDBE00860
      DO 150 I = 1,5                                                    CDBE00870
      DO 150 J = 1,5                                                    CDBE00880
      A5(I,J) = A6(I,J)                                                 CDBE00890
  150 CONTINUE                                                          CDBE00900
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                           CDBE00910
      DO 170 I = 1,4                                                    CDBE00920
      DO 170 J = 1,4                                                    CDBE00930
      A4(I,J) = A6(I,J)                                                 CDBE00940
  170 CONTINUE                                                          CDBE00950
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           CDBE00960
```

```
      DO 171 I = 1,3                                                    C3E00970
      DO 171 J = 1,3                                                    C3E00980
      A3(I,J) = A6(I,J)                                                 C3E00990
  171 CONTINUE                                                          C3E01000
      CALL LINV2F(A3,3,3,A3INV,10,WKAREA,IER)                           C3E01010
C                                                                       C3E01020
CC       COMPUTE THE CC MATRICES FOR APPROPRIATE POINTS.                C3E01030
C                                                                       C3E01040
      DO 310 K = 1,3                                                    C3E01050
      XK = DFLOAT(K)                                                    C3E01060
      H(K) = (2.0D0*XK - 1.0D0)/2.0D0                                   C3E01070
      C(K,2,1) = 1.0D0                                                  C3E01080
      C(K,3,1) = 0.0D0                                                  C3E01090
      C(K,4,1) = 0.0D0                                                  C3E01100
      C(K,3,2) = 1.0D0                                                  C3E01110
      C(K,4,2) = 0.0D0                                                  C3E01120
      C(K,4,3) = 1.0D0                                                  C3E01130
      KK = 0                                                            C3E01140
      K6 = 6                                                            C3E01150
      DO 330 I = 1,4                                                    C3E01160
      FACT = 1.0D0                                                      C3E01170
      DO 325 J = 1,K6                                                   C3E01180
      JK = J+KK                                                         C3E01190
      XJ = DFLOAT(J)                                                    C3E01200
      FACT = FACT*XJ                                                    C3E01210
      C(K,I,JK) = (H(K)**J)/FACT                                        C3E01220
  325 CONTINUE                                                          C3E01230
      KK = KK+1                                                         C3E01240
      K6 = K6-1                                                         C3E01250
  330 CONTINUE                                                          C3E01260
  310 CONTINUE                                                          C3E01270
      DO 331 K = 1,3                                                    C3E01280
      DO 331 I = 1,4                                                    C3E01290
      C(K,I,2) = C(K,I,1)                                               C3E01300
      C(K,I,1) = 0.0D0                                                  C3E01310
  331 CONTINUE                                                          C3E01320
C                                                                       C3E01330
CCCC     CONVERT THE 3 DIMENSIONAL CC MATRICES TO APPROPRIATE 2         C3E01340
C                                                                       C3E01350
CCCC     DIMENSIONAL MATRICES.                                          C3E01360
C                                                                       C3E01370
      DO 200 I = 1,4                                                    C3E01380
      DO 200 J = 1,6                                                    C3E01390
      C4X61(I,J) = C(1,I,J)                                             C3E01400
  200 CONTINUE                                                          C3E01410
      DO 201 I = 1,4                                                    C3E01420
      DO 201 J = 1,6                                                    C3E01430
      C4X62(I,J) = C(2,I,J)                                             C3E01440
```

```
  201 CONTINUE                                                          C0E01450
      DO 202 I = 1,4                                                    C0E01460
      DO 202 J = 1,6                                                    C0E01470
      C4X63(I,J) = C(3,I,J)                                             C0E01480
  202 CONTINUE                                                          C0E01490
      DO 203 I = 1,4                                                    C0E01500
      DO 203 J = 1,5                                                    C0E01510
      C4X51(I,J) = C4X61(I,J)                                           C0E01520
  203 CONTINUE                                                          C0E01530
      DO 204 I = 1,4                                                    C0E01540
      DO 204 J = 1,4                                                    C0E01550
      C4X41(I,J) = C4X61(I,J)                                           C0E01560
  204 CONTINUE                                                          C0E01570
      DO 220 I = 1,3                                                    C0E01580
      DO 220 J = 1,3                                                    C0E01590
      C3X31(I,J) = C4X61(I,J)                                           C0E01600
  220 CONTINUE                                                          C0E01610
      DO 205 I = 1,4                                                    C0E01620
      DO 205 J = 1,5                                                    C0E01630
      C4X52(I,J) = C4X62(I,J)                                           C0E01640
  205 CONTINUE                                                          C0E01650
      DO 206 I = 1,4                                                    C0E01660
      DO 206 J = 1,4                                                    C0E01670
      C4X42(I,J) = C4X62(I,J)                                           C0E01680
  206 CONTINUE                                                          C0E01690
      DO 208 I = 1,3                                                    C0E01700
      DO 208 J = 1,3                                                    C0E01710
      C3X32(I,J) = C4X62(I,J)                                           C0E01720
  208 CONTINUE                                                          C0E01730
      DO 207 I = 1,4                                                    C0E01740
      DO 207 J = 1,5                                                    C0E01750
      C4X53(I,J) = C4X63(I,J)                                           C0E01760
  207 CONTINUE                                                          C0E01770
C                                                                       C0E01780
C        PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER        C0E01790
C                                                                       C0E01800
C        CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.          C0E01810
C                                                                       C0E01820
      CALL VMULFP(C4X61,A5INV,4,6,6,4,6,C46A61,4,IER)                   C0E01830
      CALL VMULFP(C4X62,A6INV,4,6,6,4,6,C46A62,4,IER)                   C0E01840
      CALL VMULFP(C4X63,A6INV,4,6,6,4,6,C46A63,4,IER)                   C0E01850
      CALL VMULFP(C4X51,A5INV,4,5,5,4,5,C45A51,4,IER)                   C0E01860
      CALL VMULFP(C4X52,A5INV,4,5,5,4,5,C45A52,4,IER)                   C0E01870
      CALL VMULFP(C4X53,A5INV,4,5,5,4,5,C45A53,4,IER)                   C0E01880
      CALL VMULFP(C4X41,A4INV,4,4,4,4,4,C44A41,4,IER)                   C0E01890
      CALL VMULFP(C4X42,A4INV,4,4,4,4,4,C44A42,4,IER)                   C0E01900
      CALL VMULFP(C3X31,A3INV,3,3,3,3,3,C33A31,3,IER)                   C0E01910
      CALL VMULFP(C3X32,A3INV,3,3,3,3,3,C33A32,3,IER)                   C0E01920
```

```
      WRITE(2,210)                                                      CDE01930
      WRITE(2,209)                                                      CDE01940
  209 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQUCDE01950
     1IRED NEAR THE AXIS FOR N = 0.',//,1X,                             CDE01960
     2'THE COEFFICIENTS APPEAR IN THE ',                                CDE01970
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X, CDE01980
     4'THE APPLICABLE BOUNDARY CONDITIONS ARE:',//,1X,                  CDE01990
     5'DQ(0) = 0  AND  D3Q(0) = 0  FOR N = 0  AT THE AXIS.')            CDE02000
      WRITE(2,210)                                                      CDE02010
  210 FORMAT(1X,//)                                                     CDE02020
      ND = 4                                                            CDE02030
      NP = 1                                                            CDE02040
      NPTS = 6                                                          CDE02050
      WRITE(2,221) ND,NP,NPTS                                           CDE02060
  221 FORMAT(I1,1X,I1,1X,I1)                                            CDE02070
      WRITE(2,20)  (C46A61(4,J),J=1,NPTS)                               CDE02080
      NP = NP + 1                                                       CDE02090
      WRITE(2,221) ND,NP,NPTS                                           CDE02100
      WRITE(2,20)  (C46A62(4,J),J=1,NPTS)                               CDE02110
      NP = NP + 1                                                       CDE02120
      WRITE(2,221) ND,NP,NPTS                                           CDE02130
      WRITE(2,20)  (C46A63(4,J),J=1,NPTS)                               CDE02140
      ND = 3                                                            CDE02150
      NP = 1                                                            CDE02160
      NPTS = NPTS - 1                                                   CDE02170
      WRITE(2,221) ND,NP,NPTS                                           CDE02180
      WRITE(2,20)  (C45A51(3,J),J=1,NPTS)                               CDE02190
      NP = NP + 1                                                       CDE02200
      WRITE(2,221) ND,NP,NPTS                                           CDE02210
      WRITE(2,20)  (C45A52(3,J),J=1,NPTS)                               CDE02220
      NP = NP + 1                                                       CDE02230
      WRITE(2,221) ND,NP,NPTS                                           CDE02240
      WRITE(2,20)  (C45A53(3,J),J=1,NPTS)                               CDE02250
      ND = 2                                                            CDE02260
      NP = 1                                                            CDE02270
      NPTS = NPTS - 1                                                   CDE02280
      WRITE(2,221) ND,NP,NPTS                                           CDE02290
      WRITE(2,20)  (C44A41(2,J),J=1,NPTS)                               CDE02300
      NP = NP + 1                                                       CDE02310
      WRITE(2,221) ND,NP,NPTS                                           CDE02320
      WRITE(2,20)  (C44A42(2,J),J=1,NPTS)                               CDE02330
      ND = 1                                                            CDE02340
      NP = 1                                                            CDE02350
      NPTS = NPTS - 1                                                   CDE02360
      WRITE(2,221) ND,NP,NPTS                                           CDE02370
      WRITE(2,20)  (C33A31(1,J),J=1,NPTS)                               CDE02380
      NP = NP + 1                                                       CDE02390
      WRITE(2,221) ND,NP,NPTS                                           CDE02400
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```
      WRITE(2,20)  (C33A32(1,J),J=1,NPTS)                                COE02410
      ND = 9                                                             COE02420
      NP = 0                                                             COE02430
      NPTS = 0                                                           COE02440
      WRITE(2,221) ND,NP,NPTS                                            COE02450
      WRITE(2,225)                                                       COE02460
  225 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE Q(0) FOR N = 0.  TAKEN',    COE02470
     1' FROM THE TOP ROW OF THE A6INV MATRIX.')                          COE02480
      WRITE(2,20) (A6INV(1,J),J=1,6)                                     COE02490
C                                                                        COE02500
CC*********************************************************************COE02510
CC                                                                       COE02520
CC        A MODIFIED SUBPROGRAM TO COMPUTE AA MATRIX ELEMENTS FOR        COE02530
CC        THE BOUNDARY CONDITIONS NEAR THE AXIS AND THE WALL FOR         COE02540
CC        N = 0, 1 AND 6.  THE APPLICABLE BOUNDARY CONDITIONS ARE:       COE02550
CC                                                                       COE02560
CC        Q(1) = 0 AND DQ(1) = 0  FOR N = 0 AT THE WALL.                 COE02570
CC                                                                       COE02580
CC        Q(0) = VARIABLE AND DQ(0) = 0  FOR N = 1 AT THE AXIS.          COE02590
CC                                                                       COE02600
CC        Q(1) = -H(0) AND DQ(0) = 2H(0)  FOR N = 1 AT THE WALL.         COE02610
CC                                                                       COE02620
CC        Q(0) = 0 AND DQ(0) = 0  FOR N = 6 AT THE AXIS.                 COE02630
CC                                                                       COE02640
CC        Q(1) = 0 AND DQ(1) = 0  FOR N = 6 AT THE WALL.                 COE02650
CC                                                                       COE02660
CC*********************************************************************COE02670
CC                                                                       COE02680
CC        COMPUTE THE AA MATRICES AND THEIR INVERSES.                    COE02690
CC                                                                       COE02700
      DO 400 I=1,6                                                       COE02710
      FACT = 1.0D0                                                       COE02720
      XI = DFLOAT(I)                                                     COE02730
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                      COE02740
      DO 400 J=1,6                                                       COE02750
      XJ = DFLOAT(J)                                                     COE02760
      FACT = FACT*(XJ+1.0D0)                                             COE02770
      A6(I,J) = (H(I)**(J+1))/FACT                                       COE02780
  400 CONTINUE                                                           COE02790
      CALL LINV2F(A6,6,6,A6INV,10,WKAREA,IER)                            COE02800
      DO 401 I = 1,5                                                     COE02810
      DO 401 J = 1,5                                                     COE02820
      A5(I,J) = A6(I,J)                                                  COE02830
  401 CONTINUE                                                           COE02840
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                            COE02850
      DO 402 I = 1,4                                                     COE02860
```

```
      DO 402 J = 1,4                                                    CDE02890
      A4(I,J) = A6(I,J)                                                 CDE02900
  402 CONTINUE                                                          CDE02910
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           CDE02920
      DO 403 I = 1,3                                                    CDE02930
      DO 403 J = 1,3                                                    CDE02940
      A3(I,J) = A6(I,J)                                                 CDE02950
  403 CONTINUE                                                          CDE02960
      CALL LINV2F(A3,3,3,A3INV,10,WKAREA,IER)                           CDE02970
C                                                                       CDE02980
CC       COMPUTE THE CC COEFFICIENT MATRICES FOR APPROPRIATE POINTS.    CDE02990
C                                                                       CDE03000
      DO 405 K = 1,3                                                    CDE03010
      XK = DFLOAT(K)                                                    CDE03020
      H(K) = (2.0D0*XK - 1.0D0)/2.0D0                                   CDE03030
      C(K,2,1) = 1.0D0                                                  CDE03040
      C(K,3,1) = 0.0D0                                                  CDE03050
      C(K,4,1) = 0.0D0                                                  CDE03060
      C(K,3,2) = 1.0D0                                                  CDE03070
      C(K,4,2) = 0.0D0                                                  CDE03080
      C(K,4,3) = 1.0D0                                                  CDE03090
      KK = 0                                                            CDE03100
      K6 = 6                                                            CDE03110
      DO 406 I = 1,4                                                    CDE03120
      FACT = 1.0D0                                                      CDE03130
      DO 407 J = 1,K6                                                   CDE03140
      JK = J+KK                                                         CDE03150
      XJ = DFLOAT(J)                                                    CDE03160
      FACT = FACT*XJ                                                    CDE03170
      C(K,I,JK) = (H(K)**J)/FACT                                        CDE03180
  407 CONTINUE                                                          CDE03190
      KK = KK+1                                                         CDE03200
      K6 = K6-1                                                         CDE03210
  406 CONTINUE                                                          CDE03220
  405 CONTINUE                                                          CDE03230
C                                                                       CDE03240
CC       CONVERT THE 3 DIMENSIONAL CC MATRICES TO APPROPRIATE 2         CDE03250
CC                                                                      CDE03260
CC       DIMENSIONAL MATRICES.                                          CDE03270
C                                                                       CDE03280
      DO 408 I = 1,4                                                    CDE03290
      DO 408 J = 1,6                                                    CDE03300
      C4X61(I,J) = C(1,I,J)                                             CDE03310
  408 CONTINUE                                                          CDE03320
      DO 409 I = 1,4                                                    CDE03330
      DO 409 J = 1,6                                                    CDE03340
      C4X62(I,J) = C(2,I,J)                                             CDE03350
  409 CONTINUE                                                          CDE03360
```

```
      DO 410 I = 1,4                                                    CDE03370
      DO 410 J = 1,6                                                    CDE03380
      C4X63(I,J) = C(3,I,J)                                             CDE03390
  410 CONTINUE                                                          CDE03400
      DO 411 I = 1,3                                                    CDE03410
      DO 411 J = 1,5                                                    CDE03420
      C3X51(I,J) = C4X61(I,J)                                           CDE03430
  411 CONTINUE                                                          CDE03440
      DO 412 I = 1,2                                                    CDE03450
      DO 412 J = 1,4                                                    CDE03460
      C2X41(I,J) = C4X61(I,J)                                           CDE03470
  412 CONTINUE                                                          CDE03480
      DO 413 J = 1,3                                                    CDE03490
      C1X31(1,J) = C4X61(1,J)                                           CDE03500
  413 CONTINUE                                                          CDE03510
      DO 414 I = 1,3                                                    CDE03520
      DO 414 J = 1,5                                                    CDE03530
      C3X52(I,J) = C4X62(I,J)                                           CDE03540
  414 CONTINUE                                                          CDE03550
      DO 415 I = 1,2                                                    CDE03560
      DO 415 J = 1,4                                                    CDE03570
      C2X42(I,J) = C4X62(I,J)                                           CDE03580
  415 CONTINUE                                                          CDE03590
      DO 416 J = 1,3                                                    CDE03600
      C1X32(1,J) = C4X62(1,J)                                           CDE03610
  416 CONTINUE                                                          CDE03620
      DO 417 I = 1,3                                                    CDE03630
      DO 417 J = 1,5                                                    CDE03640
      C3X53(I,J) = C4X63(I,J)                                           CDE03650
  417 CONTINUE                                                          CDE03660
C                                                                       CDE03670
C        PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER        CDE03680
C                                                                       CDE03690
C        CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.          CDE03700
C                                                                       CDE03710
      CALL VMULFP(C4X61,A6INV,4,6,6,4,6,C46A61,4,IER)                   CDE03720
      CALL VMULFP(C4X62,A6INV,4,6,6,4,6,C46A62,4,IER)                   CDE03730
      CALL VMULFP(C4X63,A5INV,4,6,6,4,6,C46A63,4,IER)                   CDE03740
      CALL VMULFP(C3X51,A5INV,3,5,5,3,5,C35A51,3,IER)                   CDE03750
      CALL VMULFP(C3X52,A5INV,3,5,5,3,5,C35A52,3,IER)                   CDE03760
      CALL VMULFP(C3X53,A5INV,3,5,5,3,5,C35A53,3,IER)                   CDE03770
      CALL VMULFP(C2X41,A4INV,2,4,4,2,4,C24A41,2,IER)                   CDE03780
      CALL VMULFP(C2X42,A4INV,2,4,4,2,4,C24A42,2,IER)                   CDE03790
      CALL VMULFP(C1X31,A3INV,1,3,3,1,3,C13A31,1,IER)                   CDE03800
      CALL VMULFP(C1X32,A3INV,1,3,3,1,3,C13A32,1,IER)                   CDE03810
      WRITE(2,419)                                                      CDE03820
  419 FORMAT(1X,//)                                                     CDE03830
      WRITE(2,418)                                                      CDE03840
```

```
 418 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQUC0E03850
     1IRED NEAR THE BOUNDARIES FOR N = 0, 1 AND 6.',//,1X,                C0E03860
     2'THE COEFFICIENTS APPEAR IN THE ',                                  C0E03870
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X,   C0E03880
     4'THE APPLICABLE BOUNDARY CONDITIONS ARE:',//,1X,                    C0E03890
     5'Q(1)  = 0   AND   DQ(1)  = 0   FOR N = 0 AT THE WALL.',//,1X,      C0E03900
     6'Q(0)  = VARIABLE  AND   DQ(0) = 0   FOR N = 1 AT THE AXIS.',//,1X, C0E03910
     7'Q(1)  = -H(0)   AND   DQ(0) = 2H(0)   FOR N = 1 AT THE WALL.',//,1X,C0E03920
     8'Q(0)  = 0   AND   DQ(0)  = 0   FOR N = 6 AT THE AXIS.',//,1X,      C0E03930
     9'Q(1)  = 0   AND   DQ(1)  = 0   FOR N = 6 AT THE WALL.',//)         C0E03940
      ND = 4                                                              C0E03950
      NP = 1                                                              C0E03960
      NPTS = 6                                                            C0E03970
      WRITE(2,221) ND,NP,NPTS                                             C0E03980
      WRITE(2,20)  (C46A61(4,J),J=1,NPTS)                                 C0E03990
      NP = NP + 1                                                         C0E04000
      WRITE(2,221) ND,NP,NPTS                                             C0E04010
      WRITE(2,20)  (C46A62(4,J),J=1,NPTS)                                 C0E04020
      NP = NP + 1                                                         C0E04030
      WRITE(2,221) ND,NP,NPTS                                             C0E04040
      WRITE(2,20)  (C46A63(4,J),J=1,NPTS)                                 C0E04050
      ND = 3                                                              C0E04060
      NP = 1                                                              C0E04070
      NPTS = NPTS - 1                                                     C0E04080
      WRITE(2,221) ND,NP,NPTS                                             C0E04090
      WRITE(2,20)  (C35A51(3,J),J=1,NPTS)                                 C0E04100
      NP = NP + 1                                                         C0E04110
      WRITE(2,221) ND,NP,NPTS                                             C0E04120
      WRITE(2,20)  (C35A52(3,J),J=1,NPTS)                                 C0E04130
      NP = NP + 1                                                         C0E04140
      WRITE(2,221) ND,NP,NPTS                                             C0E04150
      WRITE(2,20)  (C35A53(3,J),J=1,NPTS)                                 C0E04160
      ND = 2                                                              C0E04170
      NP = 1                                                              C0E04180
      NPTS = NPTS - 1                                                     C0E04190
      WRITE(2,221) ND,NP,NPTS                                             C0E04200
      WRITE(2,20)  (C24A41(2,J),J=1,NPTS)                                 C0E04210
      NP = NP + 1                                                         C0E04220
      WRITE(2,221) ND,NP,NPTS                                             C0E04230
      WRITE(2,20)  (C24A42(2,J),J=1,NPTS)                                 C0E04240
      ND = 1                                                              C0E04250
      NP = 1                                                              C0E04260
      NPTS = NPTS - 1                                                     C0E04270
      WRITE(2,221) ND,NP,NPTS                                             C0E04280
      WRITE(2,20)  (C13A31(1,J),J=1,NPTS)                                 C0E04290
      NP = NP + 1                                                         C0E04300
      WRITE(2,221) ND,NP,NPTS                                             C0E04310
      WRITE(2,20)  (C13A32(1,J),J=1,NPTS)                                 C0E04320
```

```
      ND = 9                                                            COE04330
      NP = 0                                                            COE04340
      NPTS = 0                                                          COE04350
      WRITE(2,221) ND,NP,NPTS                                           COE04360
      WRITE(2,420)                                                      COE04370
  420 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE D2P(0) FOR N = 1.',        COE04380
     1 'TAKEN FROM THE FIRST ROW OF THE A6INV MATRIX.')                 COE04390
      WRITE(2,20) (A6INV(1,J),J=1,6)                                    COE04400
C*****************************************************************    COE04410
C                                                                       COE04420
C         A SUBPROGRAM USED TO COMPUTE THE NONCENTRAL FINITE            COE04430
C         DIFFERENCE COEFFICIENTS FOR THE DERIVATIVES OF P(R) AT        COE04440
C         POINTS NEAR THE BOUNDARIES.  COMPUTE THE AA MATRIX ELEMENTS   COE04450
C         FOR THE BOUNDARY CONDITIONS NEAR THE AXIS AND WALL.           COE04460
C         THE APPLICABLE BOUNDARY CONDITIONS FOR N = 1 AND 6 ARE:       COE04470
C         P(1) = I H(0), A VARIABLE, FOR N = 1 AT THE WALL.             COE04480
C         P(0) = 0  FOR N = 6 AT THE AXIS.                              COE04490
C         P(1) = 0  FOR N = 6 AT THE WALL.                              COE04500
C*****************************************************************    COE04510
C                                                                       COE04520
C         COMPUTE THE AA MATRICES AND THEIR INVERSES.                   COE04530
C                                                                       COE04540
      DO 500 I=1,5                                                      COE04550
      FACT = 1.0D0                                                      COE04560
      XI = DFLOAT(I)                                                    COE04570
      H(I) = (2.0D0*XI-1.0D0)/2.0D0                                     COE04580
      DO 500 J=1,5                                                      COE04590
      XJ = DFLOAT(J)                                                    COE04600
      FACT = FACT*XJ                                                    COE04610
      A5(I,J) =  (H(I)**J)/FACT                                         COE04620
  500 CONTINUE                                                          COE04630
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)                           COE04640
      DO 501 I = 1,4                                                    COE04650
      DO 501 J = 1,4                                                    COE04660
      A4(I,J) = A5(I,J)                                                 COE04670
  501 CONTINUE                                                          COE04680
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)                           COE04690
C                                                                       COE04700
C         COMPUTE THE CC COEFFICIENT MATRICES FOR APPROPRIATE POINTS.   COE04710
```

```
C                                                                       COE04810
      C2X51(1,1) = 1.0D0                                                COE04820
      C2X51(2,1) = 0.0D0                                                COE04830
      FACT = 1.0D0                                                      COE04840
      DO 502 J = 2,5                                                    COE04850
      XJ = DFLOAT(J)                                                    COE04860
      FACT = FACT*(XJ-1.0D0)                                            COE04870
      C2X51(1,J) = (H(1)**(J-1))/FACT                                   COE04880
  502 CONTINUE                                                          COE04890
      DO 503 J = 2,5                                                    COE04900
      JJ = J-1                                                          COE04910
      C2X51(2,J) = C2X51(1,JJ)                                          COE04920
  503 CONTINUE                                                          COE04930
      DO 504 J = 1,4                                                    COE04940
      C1X41(1,J) = C2X51(1,J)                                           COE04950
  504 CONTINUE                                                          COE04960
      C2X52(1,1) = 1.0D0                                                COE04970
      C2X52(2,1) = 0.0D0                                                COE04980
      FACT = 1.0D0                                                      COE04990
      DO 505 J = 2,5                                                    COE05000
      XJ = DFLOAT(J)                                                    COE05010
      FACT = FACT*(XJ-1.0D0)                                            COE05020
      C2X52(1,J) = (H(2)**(J-1))/FACT                                   COE05030
  505 CONTINUE                                                          COE05040
      DO 506 J = 2,5                                                    COE05050
      JJ = J-1                                                          COE05060
      C2X52(2,J) = C2X52(1,JJ)                                          COE05070
  506 CONTINUE                                                          COE05080
      DO 507 J = 1,4                                                    COE05090
      C1X42(1,J) = C2X52(1,J)                                           COE05100
  507 CONTINUE                                                          COE05110
C                                                                       COE05120
CCC       PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER       COE05130
CCC       CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.         COE05140
C                                                                       COE05150
      CALL VMULFF(C2X51,A5INV,2,5,5,2,5,C25A51,2,IER)                   COE05160
      CALL VMULFF(C2X52,A5INV,2,5,5,2,5,C25A52,2,IER)                   COE05170
      CALL VMULFF(C1X41,A4INV,1,4,4,1,4,C14A41,1,IER)                   COE05180
      CALL VMULFF(C1X42,A4INV,1,4,4,1,4,C14A42,1,IER)                   COE05190
      WRITE(2,508)                                                      COE05200
  508 FORMAT(1X,//)                                                     COE05210
      WRITE(2,509)                                                      COE05220
  509 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQU COE05230
     1IRED NEAR THE BOUNDARIES FOR N = 1 AND 6.',//,1X,                 COE05240
     2'THE COEFFICIENTS APPEAR IN THE ',                                COE05250
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X, COE05260
     4'THE APPLICABLE BOUNDARY CONDITIONS FOR N = 1 AND 6 ARE:',//,1X,  COE05270
```

```
     5'P(1) = I  H(0), A VARIABLE,  FOR N = 1 AT THE WALL.',//,1X,
     6'P(0) = 0    FOR N = 5 AT THE AXIS.',//,1X,
     7'P(1) = 0  FOR N = 6 AT THE WALL.',//)
      ND = 2
      NP = 1
      NPTS = 5
      WRITE(2,221) ND,NP,NPTS
      WRITE(2,20)  (C25A51(2,J) ,J=1,NPTS)
      NP = NP + 1
      WRITE(2,221) ND,NP,NPTS
      WRITE(2,20)  (C25A52(2,J) ,J=1,NPTS)
      ND = 1
      NP = 1
      NPTS = NPTS - 1
      WRITE(2,221) ND,NP,NPTS
      WRITE(2,20)  (C14A41(1,J) ,J=1,NPTS)
      NP = NP + 1
      WRITE(2,221) ND,NP,NPTS
      WRITE(2,20)  (C14A42(1,J) ,J=1,NPTS)
      ND = 9
      NP = 0
      NPTS = 0
      WRITE(2,221) ND,NP,NPTS
C
C*********************************************************************
C
C        A SUBPROGRAM USED TO COMPUTE THE AA MATRIX ELEMENTS FOR
C
C        THE BOUNDARY CONDITIONS NEAR THE AXIS FOR N = 1.
C
C        THE APPLICABLE BOUNDARY CONDITION IS:
C
C        DP(0) = 0  FOR N = 1 AT THE AXIS.
C
C*********************************************************************
C
      DO 600 I = 1,5
      A5(I,1) = 1.0D0
  600 CONTINUE
      DO 601 I = 1,4
      A4(I,1) = 1.0D0
  601 CONTINUE
      C2X51(1,1) = 0.0D0
      C2X52(1,1) = 0.0D0
      C1X41(1,1) = 0.0D0
      C1X42(1,1) = 0.0D0
      CALL LINV2F(A5,5,5,A5INV,10,WKAREA,IER)
      CALL LINV2F(A4,4,4,A4INV,10,WKAREA,IER)
```

```
C
CCC        PRE-MULTIPLY THE PROPER AA INVERSE MATRIX BY THE PROPER                  CDE05770
CCC                                                                                 CDE05780
CCC        CC MATRIX FOR POINTS NEAR THE BOUNDARIES AS REQUIRED.                    CDE05790
C                                                                                   CDE05800
      CALL VMULFF(C2X51,A5INV,2,5,5,2,5,C25A51,2,IER)                               CDE05810
      CALL VMULFF(C2X52,A5INV,2,5,5,2,5,C25A52,2,IER)                               CDE05820
      CALL VMULFF(C1X41,A4INV,1,4,4,1,4,C14A41,1,IER)                               CDE05830
      CALL VMULFF(C1X42,A4INV,1,4,4,1,4,C14A42,1,IER)                               CDE05840
      WRITE(2,605)                                                                  CDE05850
  605 FORMAT(1X,//)                                                                 CDE05860
      WRITE(2,606)                                                                  CDE05870
  606 FORMAT(1X,'NONCENTRAL FINITE DIFFERENCE COEFFICIENTS THAT ARE REQU            CDE05880
     1IRED NEAR THE AXIS FOR N = 1 .',//,1X,                                        CDE05890
     2'  THE COEFFICIENTS APPEAR IN THE ',                                         CDE05900
     3'FORMAT REQUIRED BY THE MAIN PROGRAM, EXCLUDING COMMENTS.',//,1X,            CDE05910
     4'THE APPLICABLE BOUNDARY CONDITION IS:',//,1X,                               CDE05920
     5'DP(0)  = 0  FOR N = 1 AT THE AXIS.',//)                                     CDE05930
      ND = 2                                                                        CDE05940
      NP = 1                                                                        CDE05950
      NPTS = 5                                                                      CDE05960
      WRITE(2,221) ND,NP,NPTS                                                       CDE05970
      WRITE(2,20)  (C25A51(2,J),J=1,NPTS)                                           CDE05980
      NP = NP + 1                                                                   CDE05990
      WRITE(2,221) ND,NP,NPTS                                                       CDE06000
      WRITE(2,20)  (C25A52(2,J),J=1,NPTS)                                           CDE06010
      ND = 1                                                                        CDE06020
      NP = 1                                                                        CDE06030
      NPTS = NPTS - 1                                                               CDE06040
      WRITE(2,221) ND,NP,NPTS                                                       CDE06050
      WRITE(2,20)  (C14A41(1,J),J=1,NPTS)                                           CDE06060
      NP = NP + 1                                                                   CDE06070
      WRITE(2,221) ND,NP,NPTS                                                       CDE06080
      WRITE(2,20)  (C14A42(1,J),J=1,NPTS)                                           CDE06090
      ND = 9                                                                        CDE06100
      NP = 0                                                                        CDE06110
      NPTS = 0                                                                      CDE06120
      WRITE(2,221) ND,NP,NPTS                                                       CDE06130
      WRITE(2,615)                                                                  CDE06140
  615 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE P(0) FOR N = 1.',                      CDE06150
     1 'TAKEN FROM THE FIRST ROW OF THE A5INV MATRIX.')                             CDE06160
      WRITE(2,20) (A5INV(1,J),J=1,5)                                                CDE06170
      WRITE(2,616)                                                                  CDE06180
  616 FORMAT(1X,'COEFFICIENTS TO APPROXIMATE D2P(0) FOR N = 1.',                    CDE06190
     1 'TAKEN FROM THE SECOND ROW OF THE A5INV MATRIX.')                            CDE06200
      WRITE(2,20) (A5INV(2,J),J=1,5)                                                CDE06210
      STOP                                                                          CDE06220
      END                                                                           CDE06230
                                                                                    CDE06240
```

```
*
*  FILENAME:   RUNDATA EXEC
*
*  THIS EXEC ALLOWS THE USER TO RUN A FORTRAN H EXTENDED PROGRAM ON
*
*  VM/CMS THAT REQUIRES INPUT DATA AND/OR PRODUCES OUTPUT DATA.
*
*  THE INPUT DATA IS STORED ON THE USER'S DISK UNDER THE SAME FILENAME
*
*  ASSIGNED TO THE MAIN PROGRAM WITH FILETYPE = <DATAIN>.  THE MAIN
*
*  PROGRAM MUST USE, READ(3,NNN) ,I.E. DEVICE CODE 3.  THE OUTPUT DATA
*
*  FROM THE FORTRAN PROGRAM WILL BE SENT TO A FILE ON THE USER'S DISK
*
*  HAVING THE SAME FILENAME WITH FILETYPE = <DATAOUT2>.  USE
*
*  WRITE(2,NNN)  , I.E. DEVICE CODE 2.  THE USER ALSO HAS THE CAPABILITY
*
*  TO LOAD AND EXECUTE PRE-COMPILED, (FILETYPE = TEXT), SUBROUTINES
*
*  THAT RESIDE ON THE USER'S DISK WHICH ARE CALLED BY THE MAIN PROGRAM.
*
*  TYPE THE COMMAND:  RUNDATA  <FN>
*
*  THE OUTPUT IS SENT TO DISK, A FILE IS CREATED AS <FN> <DATAOUT2>.
*
*  IF DEVICE CODE 6 IS also USED IN THE MAIN PROGRAM, I.E. WRITE(6,NNN),
*
*  THE OUTPUT WILL AUTOMATICALLY GO TO THE TERMINAL AS WELL.
*
FORTHX &1
FILEDEF 03 DISK &1 DATAIN (LRECL 80
FILEDEF 02 DISK &1 DATAOUT2 (BLOCK 130
FILEDEF 06 TERMINAL
LOAD &1 (START
ERASE &1 TEXT *
ERASE &1 LISTING *
ERASE LOAD MAP
&EXIT
```

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